SOME PROBLEMS IN COMBINATORIAL ANALYSIS

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "SOME PROBLEMS IN COMBINATORIAL ANALYSIS", submitted by HARVEY LESLIE ABBOTT in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

This thesis is devoted to a study of a variety of problems in combinatorial analysis.

In Chapter I some new results on a problem of Schur concerning sum-free sets of integers are obtained. Denote by $f(n)$ the largest positive integer m for which there exists some way of partitioning the integers $1, 2, \dots, m$ into n sum-free sets. It is proved that

$$f(n) > 89^{n/4} - c \log n$$

for some absolute constant c and all sufficiently large n. The methods developed can be applied successfully to other related questions.

In Chapter II some recurrence inequalities for certain of the Ramsay numbers are obtained and some new lower bounds for these numbers are derived.

In Chapter III we investigate finite families $\mathfrak{F}$ of finite sets which possess the following property: If $B \subset \bigcup \mathfrak{F}$ is such that $B \cap F \neq \Phi$ for each $F \in \mathfrak{F}$, then $B \supset F$ for some $F \in \mathfrak{F}$. Two questions of Erdös and Hajnal are considered, and a special case of Ramsay's Theorem is used to settle one of the questions.

In Chapter IV some improvements on a combinatorial theorem of Erdös and Rado are obtained and an application of the theorem to a problem in number theory is discussed.

Finally, in Chapter V, we consider the construction and enumeration of certain types of paths on the n-dimensional unit cube. In particular, a new lower bound for the number of Hamiltonian cycles on the n-cube is obtained.

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CHAPTER I

A PROBLEM OF SCHUR AND ITS GENERALIZATIONS

§1.1 A problem of Schur

A set S of integers is said to be sum-free if $a, b \in S$ implies $a + b \notin S$. We do not exclude the case where $a = b$; that is, $a \in S$ implies $2a \notin S$.

A well known theorem of I. Schur [19] states that if the integers $1, 2, \ldots, [n!\,e]$ are partitioned in an arbitrary manner into n classes, then at least one of the classes fails to be sum-free. This leads us to define $f(n)$ as the largest positive integer m for which there exists some way of partitioning the numbers $1, 2, \ldots, m$ into n sum-free sets.

It is easy to verify that $f(1) = 1$, $f(2) = 4$ and $f(3) = 13$. The determination of $f(4)$ proved to be more difficult. In 1954, H. Salié [17] published a paper in which he exhibits a distribution of the numbers $1, 2, \ldots, 43$ into four sum-free sets, thus proving that $f(4) \geq 43$. The same result was obtained by Mrs. A. Hajnal. In 1961, L. D. Baumert [2], with the aid of a high speed computer, showed that it is impossible to partition the integers $1, 2, \ldots, 45$ into four sum-free sets, and at the same time found several ways of partitioning the numbers $1, 2, \ldots, 44$ into four sum-free sets. Thus $f(4) = 44$. Since Baumert's work has not been published, we exhibit on the following page one of the ways he found of splitting the integers

A	B	C	D
1	2	4	9
3	7	6	10
5	8	13	11
15	18	20	12
17	21	22	14
19	24	23	16
26	27	25	29
28	33	30	31
40	37	32	34
42	38	39	35
44	43	41	36

Table 1. Distribution of the numbers 1, 2, ... , 44 into four sum-free sets.

1, 2, ... , 44 into four sum-free sets. The value of $f(n)$ is not known for $n > 4$ and it seems to be quite difficult to determine $f(n)$, even for $n = 5$.

The problem of finding upper and lower bounds for $f(n)$ was first considered by I. Schur who became interested in the problem through his researches on Fermat's Last Theorem. As was mentioned earlier, Schur proved

$$f(n) \leq [n!\,e] - 1 \tag{1.1.1}$$

and no improvement on this upper bound for $f(n)$ has been obtained up to the present time, although the known values of $f(n)$ indicate that (1.1.1) is not best possible. On the other hand, we have

$$f(n+1) \geq 3f(n) + 1 \tag{1.1.2}$$

and from (1.1.2) and the fact that $f(4) = 44$ it follows that, for $n \geq 4$,

$$f(n) \geq \frac{(89)3^{n-4} - 1}{2} . \tag{1.1.3}$$

An excellent exposition of the proofs of (1.1.1) and (1.1.2) can be found in the Master's thesis of H. Edgar [3].

The main result that we wish to establish in this chapter is that

$$f(n) > 89^{n/4 - c \log n} , \tag{1.1.4}$$

for some absolute constant c and all sufficiently large n.* It is clear that (1.1.4) is stronger than (1.1.3)

We find it convenient to define a function g as follows: If $f(n - 1) < k \leq f(n)$ then $g(k) = n$. $g(k)$ is thus the smallest number of sum-free sets into which the integers $1, 2, \ldots, k$ can be partitioned. It follows easily from (1.1.3) that

$$g(k) < c \log k . \tag{1.1.5}$$

In order to prove (1.1.4) we shall need the following theorem:

<u>Theorem 1.1.1</u> For all positive integers n and k,

$$f(kn + g(kf(n))) \geq (2f(n) + 1)^k - 1 . \tag{1.1.6}$$

If we set $n = 4$ in (1.1.6) and use the fact that $f(4) = 44$ we get

$$f(4k + g(44k)) \geq 89^k - 1 . \tag{1.1.7}$$

It is not difficult to see that (1.1.7) and (1.1.5) imply (1.1.4). It is very likely that (1.1.4) could be improved if the value of $f(n)$ were known for some value of $n > 4$.

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* The letter c will be used throughout the thesis to denote absolute constants. The numerical value of c will differ in different occurrences.

Proof of Theorem 1.1.1 Let $X = 2f(n) + 1$ and write the numbers $1, 2, \ldots, X^k - 1$ in base X. Call a number small if each of its digits does not exceed $f(n)$ and call a number large if at least one of its digits exceeds $f(n)$. We shall show that the small numbers can be partitioned into $g(kf(n))$ sum-free sets and the large numbers into kn sum-free sets. The theorem will then follow.

Let $A_1, A_2, \ldots, A_{g(kf(n))}$ be disjoint sum-free sets containing the integers $1, 2, \ldots, kf(n)$. Divide the small numbers into classes $B_1, B_2, \ldots, B_{g(kf(n))}$, a number being placed in class B_j if the sum of its digits belongs to A_j. This can be done since the sum of the digits of a small number does not exceed $kf(n)$. It is not difficult to see that each B_j is sum-free.

Divide the large numbers into k classes $C_1, C_2, \ldots, C_k$ by placing $a = a_1 + a_2X + \ldots + a_kX^{k-1}$ in class C_j if $a_i \leq f(n)$ for $i = 1, 2, \ldots, j - 1$ and $a_j \geq f(n) + 1$. Next divide each C_j into n sum-free sets in the following way: Let $D_1, D_2, \ldots, D_n$ be disjoint sum-free sets containing the integers $1, 2, \ldots, f(n)$ and split C_j into n sets $D_{j_1}, D_{j_2}, \ldots, D_{j_n}$ by placing the number $a = a_1 + a_2X + \ldots + a_kX^{k-1} \in C_j$ in D_{j_ℓ} if $a_j \equiv -u (\mathrm{mod}\ X)$ for some $u \in D_\ell$. Since a_j is one of the numbers $f(n) + 1, f(n) + 2, \ldots, 2f(n)$ exactly one such u can be found. It remains to be shown that D_{j_ℓ} is sum-free. Suppose we can find a, b, c in D_{j_ℓ} such that $a + b = c$. We have

$$a = a_1 + a_2X + \dots + a_kX^{k-1}$$

$$b = b_1 + b_2X + \dots + b_kX^{k-1}$$

$$c = c_1 + c_2X + \dots + C_kX^{k-1}$$

where $a_i, b_i, c_i \leq f(n)$ for $i = 1, 2, \dots, j-1$, $a_j, b_j, c_j \geq f(n) + 1$ and

$$a_j \equiv -u(\text{mod } X)$$

$$b_j \equiv -v(\text{mod } X)$$

$$c_j \equiv -w(\text{mod } X)$$

where $u, v, w \in D_\ell$. Since $a_j + b_j = X + c_j$, it follows that $u + v \equiv w(\text{mod } X)$. Since $u, v, w \leq f(n)$, we must have $u + v = w$. This contradicts the fact that D_ℓ is sum-free. The large numbers have therefore been divided into kn sum-free sets and the proof of the theorem is complete.

Using (1.1.6) it is not difficult to prove that $\ell = \lim\limits_{n \to \infty} f(n)^{1/n}$ exists, but we cannot decide whether the limit is finite or infinite although it is likely that $\ell = \infty$.

§1.2 Some related questions

In this section we discuss some questions related to Schur's problem and mention some unsolved problems.

Denote by $F(n)$ the largest positive integer m such that the integers $1, 2, \ldots, m$ can be partitioned into n classes in wuch a way that if a and b are in the same class and $a \neq b$ then $a + b$ does not belong to this class. It is known, and easy to verify, that $F(1) = 2$, $F(2) = 7$, and $F(3) = 22$. Thus it appears that $F(n)$ increases more rapidly than $f(n)$. However, it seems that the argument used to prove (1.1.6) cannot be used to prove an analogous inequality for F. In fact, it is not even known whether the analogue of (1.1.2) holds for F. The only inequality that one has relating $F(n + 1)$ and $F(n)$ is

$$F(n + 1) \geq 2F(n) + 2.$$

It is clear, of course, that

$$F(n) \geq f(n)$$

and hence, in view of (1.1.4), that

$$F(n) > 89^{n/4 - c \log n}.$$

It is known [3] that $F(n) \leq 2[n!\, e]$. This suggests that possibly $F(n) \leq 2f(n)$, but this has not been proved nor disproved.

If n and m are positive integers, denote by $f(m, n)$ the largest positive integer q such that the integers $m, m + 1, \ldots, m + q$ can be partitioned into n sum-free sets. It is clear that

$$f(1, n) = f(n) - 1.$$

We now prove

$$(1.2.1) \qquad f(m, n) \leq mf(n) - 1.$$

For suppose there exists some way of splitting the numbers $m, m+1, \ldots, m + mf(n)$ into n sum-free sets. Then one would have a partitioning of the integers $m, 2m, \ldots, m(f(n) + 1)$ and hence also the integers $1, 2, \ldots, f(n) + 1$ into n sum-free sets. This contradiction establishes (1.2.1). From (1.2.1) and the fact that $f(n) \leq [n!\, e] - 1$ we get

$$(1.2.2) \qquad f(m, n) \leq m[n!\, e] - m - 1.$$

Equality holds in (1.2.1) for $n = 1, 2, 3$. In fact, for $n = 1$ we can take

$$S_1 = \{m, m+1, \ldots, 2m-1\}.$$

For $n = 2$ we can take

$$S_1 = \{m, m+1, \ldots, 2m-1\} \cup \{4m, 4m+1, \ldots, 5m-1\}$$

$$S_2 = \{2m, 2m+1, \ldots, 4m-1\},$$

and for $n = 3$ we can take

$$S_1 = \{m, \ldots, 2m-1\} \cup \{4m, \ldots, 5m-1\} \cup \{10m, \ldots, 11m-1\} \cup \{13m, \ldots, 14m-1\}$$

$$S_2 = \{2m, \ldots, 4m-1\} \cup \{11m, \ldots, 13m-1\}$$

$$S_3 = \{5m, \ldots, 10m-1\}.$$

It is not known whether equality holds in (1.2.1) for any larger values of n.

We proceed to find a lower bound for $f(m, n)$. We prove first

(1.2.3) $$f(m, n+1) \geq 3f(m, n) + m + 2.$$

Suppose that the numbers $m, m+1, \ldots, m + f(m, n)$ have already been distributed into n sum-free sets $C_1, C_2, \ldots, C_n$. Into C_{n+1} we place the numbers $m + f(m, n) + 1, m + f(m, n) + 2, \ldots, 2m + 2f(m, n) + 1$. It is clear that C_{n+1} is sum-free. Now adjoin to the sets $C_1, C_2, \ldots, C_n$ the numbers $2m + 2f(m, n) + 2$, $2m + 2f(m, n) + 3, \ldots,$ $2m + 3f(m, n) + 2$, the number $2m + 2f(m, n) + 2 + t$ being adjoined to C_j if $m + t \in C_j$. Denote the new sets by $C_1', C_2', \ldots, C_n'$. It remains to be shown that each C_j' is sum-free. Suppose there exists integers $x, y, z \in C_j'$ such that $x + y = z$. Let us take the case where $x \in C_j$ and $y, z \in C_j' \sim C_j$. Then we have

$$\begin{aligned} x &= m + t, && 0 \leq t \leq f(m, n) \\ y &= 2m + 2f(m, n) + 2 + s, && 0 \leq s \leq f(m, n), \quad m + s \in C_j \\ z &= 2m + 2f(m, n) + 2 + r, && 0 \leq r \leq f(m, n), \quad m + r \in C_j. \end{aligned}$$

Then

$$\begin{aligned} x + y &= 3m + 2f(m, n) + 2 + t + s \\ &= z \\ &= 2m + 2f(m, n) + 2 + r. \end{aligned}$$

Hence

$$m + t + s = r.$$

Equivalently,

$$(m + t) + (m + s) = (m + r).$$

But $m + t, m + s, m + r \in C_j$ and this contradicts the fact that C_j is sum-free. The other possibilities can be disposed of in a similar fashion.

From (1.2.3) it follows by induction on m that

$$f(m, n) \geq \frac{m3^n - m - 2}{2} .$$

Now one can ask the following question, the answer to which is not known: Does there exist a constant $c > 3$ such that

$$f(m, n) > mc^n$$

for <u>all</u> m and all sufficiently large n?

§1.3 Generalizations of Schur's Problem

As was observed by R. Rado [14], the problem of Schur is a special case of a more general problem. Consider the following equation in m unknowns $x_1, x_2, \ldots, x_m$:

(1.3.1) $$\sum_{i=1}^{m} a_i x_i = 0 ,$$

where the a's are non-zero integers. (1.3.1) is said by Rado to be <u>n-fold regular</u> if there exists a non-negative integer $f(n)$, which we take to be minimal, such that if the integers $1, 2, \ldots, f(n) + 1$ are partitioned in any way into n classes, then at least one of the

classes contains a solution of (1.3.1). (1.3.1) is said to be <u>regular</u> if it is n-fold regular for every positive integer n.

One of the main results which Rado establishes is the following criterion giving necessary and sufficient conditions for an equation to be regular: (1.3.1) is regular if and only if some subset of the coefficients has zero sum. Thus the equation $x_1 + x_2 - x_3 = 0$ is regular and it is easy to see that the problem of Schur consists of finding bounds for $f(n)$ for the equation $x_1 + x_2 - x_3 = 0$. The problem of finding lower bounds for $f(n)$ for a number of regular equations was considered by H. Salié [17].

Write (1.3.1) in the form

$$\sum_{i=1}^{\ell} a_i x_i = \sum_{i=\ell+1}^{m} a_i x_i \tag{1.3.2}$$

where the a's are positive integers. Suppose (1.3.2) is regular. Then we can assume that $A = a_1 + a_2 + \dots + a_\ell > a_{\ell+1} + \dots + a_m$. Let g be a function defined as follows: if $f(n-1) < k \leq f(n)$, then $g(k) = n$. Suppose, in addition, that for some n it is possible to partition the integers $1, 2, \dots, (A-1)f(n)$ into classes, no class containing a solution of any of the equations

$$\sum_{i=1}^{\ell} a_i x_i = \sum_{i=\ell+1}^{m} a_i x_i + \mu(Af(n)+1), \qquad \mu = 0, \pm 1, \dots, \pm(A-2)$$

and let $h(n)$ denote the least number of classes required. Then we can prove

Theorem 1.3.1 For all positive integers k and any positive integer n for which $h(n)$ is defined,

$$f(kh(n) + g(kf(n))) \geq (Af(n) + 1)^k - 1. \tag{1.3.3}$$

The proof of Theorem 1.3.1 is only slightly more complicated than the proof of Theorem 1.1.1 and we shall not give all of the details. To prove (1.3.3), write the numbers $1, 2, \ldots, (Af(n) + 1)^k - 1$ in base $Af(n) + 1$. Call a number small if each of its digits does not exceed $f(n)$ and call a number large if at least one of its digits exceeds $f(n)$. Then one can partition the small numbers into $g(kf(n))$ classes and the large numbers into $kh(n)$ classes, no class containing a solution of (1.3.2).

Theorem 1.3.1 can be applied successfully to several regular equations. If the equation in question is $x + y = z$, then (1.3.3) reduces to (1.1.6). We mention some other examples.

Consider the equation

$$2x + y = 2z \tag{1.3.4}$$

which is regular by Rado's criterion. H. Salié [17] proved that

$$f(n) \geq 2^n - 1 \tag{1.3.5}$$

by splitting the integers $1, 2, \ldots, 2^n - 1$ into classes $C_1, C_2, \ldots, C_n$, a number being placed in class C_j if it is of the form $(2t + 1)2^{j-1}$. Salié also observed that equality holds in (1.3.5) for $n = 1, 2$, but that $f(3) = 13$. That $f(3) \geq 13$ can be seen from the

following partitioning of the integers $1, 2, \ldots, 13$ into three classes none of which contains a solution of (1.3.4):

$$A = \{1, 6, 7, 8, 13\}$$
$$B = \{2, 5, 9, 12\}$$
$$C = \{3, 4, 10, 11\} .$$

By trial and error one can show that $f(3) < 14$ and thus $f(3) = 13$. The numbers $1, 2, \ldots, 26 = (A - 1)f(3)$ can be partitioned into five classes, no class containing a solution of any of the equations

$$2x + y = 2z$$
$$2x + y = 2z + 40$$
$$2x + y = 2z - 40$$

in the following way:

$$A = \{1, 6, 7, 8, 13\}$$
$$B = \{2, 5, 9, 12\}$$
$$C = \{3, 4, 10, 11\}$$
$$D = \{14, 15, 16, 17, 18, 19, 20\}$$
$$E = \{21, 22, 23, 24, 25, 26\} .$$

Thus $h(3) = 5$. From (1.3.3) it follows that

$$f(5k + g(13k) \geq 40^k - 1$$

and this implies that, for n sufficiently large,

(1.3.6) $$f(n) > 40^{n/5} - c \log n .$$

It is clear that (1.3.6) is stronger than (1.3.5).

Consider next the equation

$$x + y + z = 2w. \tag{1.3.7}$$

Salié proved that

$$f(n) \geq 2^n - 1 \tag{1.3.8}$$

and in fact the same argument used to prove (1.3.5) can be used to prove (1.3.8). A better lower bound for $f(n)$ can be obtained using Theorem 1.3.1. It is easy to verify that $f(2) = 3$ and that $h(2) = 3$. Thus we get from (1.3.3)

$$f(3k + g(3k)) \geq 9^k - 1$$

and hence, if n is sufficiently large,

$$f(n) > 9^{n/3 - c \log n}. \tag{1.3.9}$$

In conclusion, we remark that the methods developed in this chapter do not seem to apply to the equation $x + y = 2z$, with the restriction $x \neq y$. The best result that is known in connection with this problem was obtained by L. Moser [13] who proved that it is possible to partition the integers $1, 2, \ldots, [c^{(\log k)^2}]$ into k classes, no class containing a solution of $x + y = 2z$, $x \neq y$.

CHAPTER II

SOME INEQUALITIES FOR THE RAMSAY NUMBERS

§2.1 Ramsay's Theorem

In 1930, F. P. Ramsay published a paper [16] containing a combinatorial theorem which is now well known and which has given rise to an extensive literature. The theorem can be formulated as follows:

Ramsay's Theorem: Let n, k and r be positive integers with $k \geq r$. Then there exists a positive integer $N(n, k, r)$, which we take to be minimal, such that if $s \geq N(n, k, r)$, S is a set of s elements, and the collection of subsets of S with r elements is partitioned in an arbitrary manner into n classes, then there is some subset K of S with k elements such that the subsets of K with r elements all belong to the same class.

In this chapter we shall be concerned only with the case where $r = 2$. For notational convenience denote $N(n, k, 2)$ by $\ell(n, k)$. This special case of Ramsay's Theorem can be formulated in the language of graph theory in the following way: If G is a complete graph on $\ell \geq \ell(n, k)$ vertices and if each edge of G is colored in any one of n colors, then there will result a complete subgraph of G on k vertices, all of whose edges have the same color.

In what follows we shall often refer to "a complete subgraph on k vertices, all of whose edges have the same color" as "a complete monochromatic subgraph on k vertices" or as "a complete monochromatic k-gon".

The problem of determining $\ell(n, k)$ appears to be a very difficult one. The value of $\ell(n, k)$ is not known for $n > 4$ and even for $n \leq 4$ its value is known only for small values of k (except when $n = 1$, in which case we have $\ell(1, k) = k$ for all k). It is not surprising that even less is known concerning exact values of $N(n, k, r)$ with $r > 2$. In addition, very little is known as to what is the order of magnitude of $\ell(n, k)$ and all existing upper and lower bounds for $\ell(n, k)$ are quite far apart. In this chapter some progress is made towards narrowing the gap.

First, we discuss briefly some of the known results. G. Szekeres [21] proved that

$$\ell(2, k) \leq \binom{2k-2}{k-1} \tag{2.1.1}$$

and his argument can be generalized to prove that

$$\ell(n, k) \leq \frac{(nk-n)!}{((k-1)!)^n} . \tag{2.1.2}$$

Both of these results are also discussed in the paper of R. E. Greenwood and A. M. Gleason [12]. The only other upper bound for $\ell(n, k)$ which appears in the literature is the following one obtained by T. Skolem [20]:

$$\ell(n, k) \leq \frac{n^{kn-k+2}}{n-1} . \tag{2.1.3}$$

It is not difficult to see that there is very little difference between (2.1.2) and (2.1.3). If we set $k = 3$ in (2.1.3) we get

$$\ell(n, 3) \leq \frac{n^{3n}}{n(n-1)} .$$

However, in this case it is known that

$$\ell(n, 3) \leq [n! e] + 1,$$

and a proof of this result can be found in [12].

We turn our attention next to the known lower bounds for $\ell(n, k)$. P. Erdős [4] proved, by probabilistic methods, that

$$\binom{\ell(2, k)}{k} \geq 2^{\binom{k}{2} - 1} \tag{2.1.4}$$

and this implies

$$\ell(2, k) > ck2^{k/2} . \tag{2.1.5}$$

A proof of (2.1.4) which is different from that of Erdős will be given in Chapter III of this thesis. The argument used by Erdős to prove (2.1.4) can be used to prove

$$\binom{\ell(n, k)}{k} \geq n^{\binom{k}{2} - 1} . \tag{2.1.6}$$

It is not difficult to check that the lower bound for $\ell(n, k)$ afforded by (2.1.6) is approximately $kn^{k/2}$. In the next section we shall prove a theorem which, when combined with (2.1.5), will yield a much better lower bound for $\ell(n, k)$ for large values of n.

In concluding this section, we mention a problem which was raised by Erdös [4]. The sequence $\{\ell(2, k)^{1/k}\}$ is bounded, and in fact we have from (2.1.1) and (2.1.2)

$$\sqrt{2} \leq \liminf_{k \to \infty} \ell(2, k)^{1/k} \leq \limsup_{k \to \infty} \ell(2, k)^{1/k} \leq 4.$$

Now one can ask: does $\ell = \lim_{k \to \infty} \ell(2, k)^{1/k}$ exist? It seems likely that the answer is yes, but this question has not been settled.

§2.2 A recurrence inequality for $\ell(n, k)$ for fixed k

In this section we obtain a recurrence inequality for $\ell(n, k)$ from which we deduce a new lower bound for $\ell(n, k)$.

Theorem 2.2.1 For all positive integers n and m,

$$\ell(n + m, k) \geq (\ell(n, k) - 1)(\ell(m, k) - 1) + 1. \tag{2.2.1}$$

Proof: For notational convenience set

$$h(n, k) = \ell(n, k) - 1.$$

Then we have to prove

$$h(n + m, k) \geq h(n, k)h(m, k). \tag{2.2.2}$$

Suppose there are $n + m$ colors $C_1, C_2, \ldots, C_n, C_{n+1}, \ldots, C_{n+m}$ available. Let G be a complete graph with vertices P_i, $i = 1, 2, \ldots, h(n, k)$. Color the edges of G in n colors $C_1, C_2, \ldots, C_n$ in such a way that there does not result a complete monochromatic subgraph on k vertices. This is possible, by the definition of $h(n, k)$. For $i = 1, 2, \ldots, h(n, k)$ let G_i be a complete graph with vertices P_{ij}, $j = 1, 2, \ldots, h(m, k)$. Color the edges of each G_i in m colors $C_{n+1}, C_{n+2}, \ldots, C_{n+m}$ in such a way

that there does not result in any G_i any complete monochromatic subgraph on k vertices. Let H be the complete graph with vertices P_{ij}, $i = 1, 2, \ldots, h(n, k)$, $j = 1, 2, \ldots, h(m, k)$. Color the edges of H in the $n + m$ colors $C_1, C_2, \ldots, C_n, C_{n+1}, \ldots, C_{n+m}$ as follows: Consider the vertices P_{st} and P_{uv}. If $s = u$, then P_{st} and P_{uv} are vertices of G_s, and the edge joining the vertices is colored the same as it is in G_s. If $s \neq u$, the edge is to be colored the same as the edge joining P_s and P_u in G. It is clear that H has $h(n, k)h(m, k)$ vertices and in order for H to contain a complete subgraph on k vertices, all of whose edges have the same color, then either G contains such a subgraph or one of the G_i contains such a subgraph. Since this is not the case, the proof of the theorem is complete.

From (2.2.2) it follows that

$$
\begin{aligned}
h(2n, k) &\geq h(2, k)^n \\
h(2n + 1, k) &\geq h(1, k)h(2, k)^n
\end{aligned}
\tag{2.2.3}
$$

and it is not difficult to see that (2.2.3), together with (2.1.5), yields a lower bound for $\ell(n, k)$ which is approximately $k^{n/2}(2^{n/2})^{k/2}$. This is clearly better than the lower bound given by (2.1.6).

§2.3 Lower bounds for $\ell(n, k)$ for "small" values of k

We discuss next some lower bounds for $\ell(n, k)$ for $k = 3, 5$.

A lower bound for $\ell(n, 3)$ In the case $k = 3$, it is known and easy to prove that $\ell(2, 3) = 6$. R. E. Greenwood and A. M. Gleason [12] proved that $\ell(3, 3) = 17$. That $\ell(3, 3) \leq 17$ follows from their result

$$(2.3.1) \qquad \ell(n, 3) \leq [n!\, e] + 1$$

and they showed how to color the edges of a complete graph on 16 vertices in three colors so that the resulting configuration contains no triangle with all of its edges having the same color. According to Greenwood [11] it is now known that $\ell(4, 3) = 66$, but this result has not yet been published. It is interesting to observe that equality holds in (2.3.1) for $n = 1, 2, 3, 4$.

A lower bound for $\ell(n, 3)$ can be obtained using (2.2.1). In fact, one can easily show that

$$\ell(n, 3) > c(65)^{n/4}.$$

However, a better lower bound can be obtained by a different argument. We prove

Theorem 2.3.1 Let f be the function defined in the problem of Schur; that is, $f(n)$ is the largest positive integer such that the numbers $1, 2, \ldots, f(n)$ can be partitioned into n sum-free sets. Then we have

$$(2.3.2) \qquad \ell(n, 3) \geq f(n) + 2.$$

It follows immediately from (2.3.2) and (1.1.4) that

(2.3.3) $$\ell(n, 3) > 89^{n/4 - c \log n}.$$

It is also interesting to observe that (2.3.1) and (2.3.2) afford an independent proof of (1.1.1).

<u>Proof of Theorem 2.3.1</u> Let $A_1, A_2, \ldots, A_n$ be disjoint sum-free sets containing the integers $1, 2, \ldots, f(n)$. Let G be a complete graph with $f(n) + 1$ vertices $P_0, P_1, \ldots, P_{f(n)}$. We show that it is possible to color the edges of G in n colors $C_1, C_2, \ldots, C_n$ in such a way that the resulting configuration contains no monochromatic triangle. Color the edge joining P_s and P_t color C_j if $|s - t| \in A_j$. Suppose there results a triangle with vertices P_s, P_t, P_r all of whose edges are colored C_j. There is no loss of generality in assuming $s > t > r$. Then we must have $s - t$, $s - r$, $t - r \in A_j$. However, $(s - t) + (t - r) = (s - r)$ and this contradicts the fact that A_j is sum-free. The proof of the theorem is complete.

More generally we can prove that if $f_k(n)$ denotes the largest positive integer such that one can partition the integers $1, 2, \ldots, f_k(n)$ into n classes, no class containing a solution of the equation $x_1 + x_2 + \ldots + x_{k-1} = x_k$, then

(2.3.4) $$\ell(n, k) \geq f_k(n) + 2.$$

Thus any lower bound for $f_k(n)$ automatically yields a lower bound for $\ell(n, k)$. At the present time, however, the best lower bound that is known for $f_k(n)$ is

$$f_k(n) \geq k^n - k^{n-1} - k^{n-2} - \dots - k - 2,$$

and hence (2.3.4) yields only

$$\ell(n, k) \geq k^n - k^{n-1} - k^{n-2} - \dots - k,$$

which is very poor for large values of n and k.

<u>A lower bound for $\ell(n, 5)$</u> That $\ell(2, 5) \geq 38$ can be established in the following way: Let G be a complete graph with vertices $P_1, P_2, \dots, P_{37}$. Color the edges of G in two colors C_1 and C_2 by coloring the edge joining P_s and P_t color C_1 if $s - t$ is a quadratic residue of 37 and color C_2 if $s - t$ is a quadratic non-residue of 37. Then it is not difficult to check that G contains no complete monochromatic pentagon. It now follows from Theorem 2.2.1 that

$$\ell(n, 5) > c(37)^{n/2}.$$

A much better lower bound for $\ell(n, 5)$ can be obtained from the following theorem.

<u>Theorem 2.3.2</u> For all integers $a, b \geq 2$,

$$\ell(n, ab - a - b + 2) \geq (\ell(n, a) - 1)(\ell(n, b) - 1) + 1. \tag{2.3.5}$$

<u>Proof:</u> It is sufficient to prove

$$h(n, ab - a - b + 2) \geq h(n, a)h(n, b). \tag{2.3.6}$$

Let G be a complete graph with vertices $P_1, P_2, \dots, P_{h(n,b)}$. Color the edges of G in n colors $C_1, C_2, \dots, C_n$ in such a way

that there does not result a complete subgraph of G on b vertices, all of whose edges have the same color. For $i = 1, 2, \dots, h(n, b)$ let G_i be a complete graph with vertices P_{ij}, $j = 1, 2, \dots, h(n, a)$. Color the edges of each G_i in the n colors $C_1, C_2, \dots, C_n$ in such a way that there does not result in any G_i a complete monochromatic a-gon.

Let H be a complete graph with vertices P_{ij}, $i = 1, 2, \dots, h(n, a)$, $j = 1, 2, \dots, h(n, b)$. Consider the edge E joining P_{st} and P_{uv}. If $s = u$, then E is an edge of G_s and is colored the same as it is colored in G_s. If $s \neq u$, E is to be colored the same as the edge joining P_s and P_u in G. Suppose H has a complete subgraph K on $ab - a - b + 2$ vertices all of whose edges have the same color, say C_1. We consider two cases.

<u>Case 1</u> The vertices of K are distributed over at least b of the graphs $G_1, G_2, \dots, G_{h(n,b)}$. Then it is not difficult to see that G must contain a complete subgraph on b vertices all of whose edges are colored C_1. This is a contradiction.

<u>Case 2</u> The vertices of K are distributed over $t \leq b - 1$ of the graphs $G_1, G_2, \dots, G_{h(n,b)}$, say $G_{i_1}, G_{i_2}, \dots, G_{i_t}$. Then at least a of the vertices of K belong to G_{i_j} for some j, $1 \leq j \leq t$, since otherwise the number of vertices of K would not exceed $t(a - 1) \leq ab - a - b + 1$. This implies that G_{i_j} contains a complete subgraph on a vertices all of whose edges are colored C_1. This is also a contradiction. The theorem follows.

If we take $a = b = 3$ in (2.3.5) we get

$$\ell(n, 5) \geq (\ell(n, 3) - 1)^2 + 1.$$

Thus, in view of (2.3.3),

$$\ell(n, 5) > 89^{n/2 - c \log n} .$$

Lower bounds for $\ell(n, k)$ for other small values of k can be obtained from (2.3.5) by assigning values to a and b, but we shall not mention any of these here.

Note that if we set $n = 2$ in (2.3.5) we get

$$h(2, ab - a - b + 2) \geq h(2, a)h(2, b)$$

and this implies

$$h(2, ab) \geq h(2, a)h(2, b). \tag{2.3.7}$$

One can now ask whether or not there exists an inequality relating $h(2, a + b)$ and $h(2, a)h(2, b)$. If one could prove that $h(2, a + b) \geq h(2, a)h(2, b)$, say, then the question of the existence of $\ell = \lim_{k \to \infty} \ell(2, k)^{1/k}$ would be answered in the affirmative.

CHAPTER III

ON A PROPERTY OF FAMILIES OF SETS

§3.1 Property $\mathcal{B}$

A family $\mathcal{F}$ of sets is said to possess property $\mathcal{B}$ if there exists a set $B \subset \cup\mathcal{F}$ such that $B \cap F \neq \Phi$ and $F \not\subset B$ for each $F \in \mathcal{F}$.

Several well known theorems are related in some sense to property $\mathcal{B}$. For example, a theorem of van der Waerden [22] states that to each positive integer $k \geq 3$ there corresponds a positive integer $w(k)$, which we take to be minimal, such that if the integers $1, 2, \ldots, w(k)$ are partitioned in an arbitrary manner into two classes, at least one class contains an arithmetic progression of k terms. As was pointed out by Erdös [5], van der Waerden's theorem can be formulated as follows: Let $\mathcal{F}_{k,w}$ denote the family of all arithmetic progressions of k terms contained in the interval $[1, w]$. Then there exists a least positive integer $w(k)$ such that if $w \geq w(k)$, then $\mathcal{F}_{k,w}$ does not possess property $\mathcal{B}$. Later in this chapter we shall exhibit a relationship between Ramsay's Theorem and property $\mathcal{B}$.

There have been several papers written about families of sets which do or do not possess property $\mathcal{B}$. The first papers on the subject were devoted primarily to the study of infinite families of infinite sets. However, as was pointed out by P. Erdös and A. Hajnal [8],

several interesting, and perhaps deep, questions can be asked about finite families of finite sets. It is to such questions that we devote our attention in this chapter.

§3.2 A Problem of Erdös and Hajnal

Erdös and Hajnal [8] observed that if S is a set of $2n - 1$ elements and $\mathcal{F}$ is the family of subsets of S with n elements, then $\mathcal{F}$ does not possess property $\mathcal{B}$. To see this let $B \subset \cup \mathcal{F}$ be such that $B \cap F \neq \Phi$ for each $F \in \mathcal{F}$. Then $|B| \geq n$, since otherwise B would have empty intersection with at least one member of $\mathcal{F}$. But $|B| \geq n$ implies that B contains some member of $\mathcal{F}$. This observation led them to the following question: What is the smallest integer $m(n)$ for which there exists a family $\mathcal{F}_n$ of sets $A_1, A_2, \ldots, A_{m(n)}$ such that $|A_i| = n$ for $i = 1, 2, \ldots, m(n)$ and which does not possess property $\mathcal{B}$?

It follows from the above argument that

$$m(n) \leq \binom{2n-1}{n} < 4^n . \tag{3.2.1}$$

Erdös and Hajnal also observed that $m(1) = 1$, $m(2) = 3$ and $m(3) = 7$. That $m(2) \leq 3$ and $m(3) \leq 7$ follows from the fact that the families

$$\mathcal{F}_2 = \{(1, 2), (1, 3), (2, 3)\}$$

and

$$\mathcal{F}_3 = \{(1,2,3), (1,4,5), (1,6,7), (2,4,6), (2,5,7), (3,4,7), (3,5,6)\}$$

do not possess property $\mathcal{B}$. By trial and error it can be shown that $m(2) > 2$ and $m(3) > 6$. The value of $m(n)$ is not known for $n \geq 4$ and it does not seem easy to determine $m(n)$, even for $n = 4$.

Erdös [5] proved that, for all $n \geq 2$,

$$m(n) > 2^{n-1}, \tag{3.2.2}$$

and by more complicated arguments he was able to prove that

$$m(n) > (1 - \epsilon)2^n \log 2$$

for every $\epsilon > 0$ and $n \geq n_0(\epsilon)$. The best lower bound that has been established for $m(n)$ up to the present time is the following one due to W. Schmidt [18]:

$$m(n) > 2^n\left(\frac{n}{n + 4}\right) .$$

However, this result is not much better than (3.2.2).

One of the main results that we wish to present in this chapter is the derivation of a better upper bound for $m(n)$ than the one given by (3.2.1). We prove

$$m(n) < (\sqrt{7} + \epsilon)^n \tag{3.2.3}$$

for every $\epsilon > 0$ and $n \geq n_0(\epsilon)$. We shall need the following theorems.

<u>Theorem 3.2.1</u> For all positive integers a and b,

$$m(ab) \leq m(a)m(b)^a. \tag{3.2.4}$$

<u>Proof</u>: Let $\mathcal{F}_a = \{A_1, A_2, \ldots, A_{m(a)}\}$ be a family of sets which does not possess property $\mathcal{B}$. As the notation implies $|A_i| = a$ for $i = 1, 2, \ldots, m(a)$. Let $\bigcup_{i=1}^{m(a)} A_i = \{x_1, x_2, \ldots, x_\ell\}$, and for $j = 1, 2, \ldots, \ell$ let $\mathcal{F}_b^j = \{B_1^j, B_2^j, \ldots, B_{m(b)}^j\}$ be families of sets which do not possess property $\mathcal{B}$. We assume that the sets in $\mathcal{F}_b^j$ are disjoint from the sets in $\mathcal{F}_b^k$ if $j \neq k$. Choose $A_i = \{x_{i_1}, x_{i_2}, \ldots, x_{i_a}\} \in \mathcal{F}_a$ and from each of the families $\mathcal{F}_b^{i_1}, \mathcal{F}_b^{i_2}, \ldots, \mathcal{F}_b^{i_a}$ pick one B. The union of these B's is a set consisting of ab elements. Let $\mathcal{F}$ be the family of all possible sets constructed in this way. It is clear that the number of sets in $\mathcal{F}$ is $m(a)m(b)^a$. It remains to be shown that $\mathcal{F}$ does not possess property $\mathcal{B}$. Let $B \subset \cup\mathcal{F}$ be such that $B \cap F \neq \Phi$ for each $F \in \mathcal{F}$. There are two cases to be considered.

<u>Case 1</u> There exists $A_i = \{x_{i_1}, x_{i_2}, \ldots, x_{i_a}\} \in \mathcal{F}_a$ such that B has non empty intersection with each member of each of the families $\mathcal{F}_b^{i_1}, \mathcal{F}_b^{i_2}, \ldots, \mathcal{F}_b^{i_a}$. Then B contains at least one member of each of these families and hence contains a member of $\mathcal{F}$.

<u>Case 2</u> In each $A_i \in \mathcal{F}_a$ there is an element which we denote by $x_{A_i'}$ such that B has empty intersection with one of the sets in $\mathcal{F}_b^{A_i'}$. Let $T = \{x_{A_i'} : i = 1, 2, \ldots, m(a)\}$. Then $T \cap A \neq \Phi$ for each $A \in \mathcal{F}_a$. Thus $T \supset A_j = \{x_{j_i}, x_{j_2}, \ldots, x_{j_a}\}$ for some j, $1 \leq j \leq m(a)$. It follows that B has empty intersection with at least one of the sets in each of the families $\mathcal{F}_b^{j_1}, \mathcal{F}_b^{j_2}, \ldots, \mathcal{F}_b^{j_a}$ and

hence has empty intersection with at least one $F \in \mathcal{F}$. This contradicts the fact that $B \cap F \neq \Phi$ for each $F \in \mathcal{F}$. The proof of the theorem is complete.

It is obvious that we have the following

<u>Corollary</u> $m(ab) \leq \min\{m(a)m(b)^a, \; m(b)m(a)^b\}$.

<u>Theorem 3.2.2</u> $m(n+1) \geq m(n)$.

<u>Proof</u>: Let $\mathcal{F} = \{A_1, A_2, \ldots, A_{m(n+1)}\}$ be a family of sets, each set with $n+1$ elements, which does not possess property $\mathcal{B}$. Let $\mathcal{F}' = \{A_1', A_2', \ldots, A_{m(n+1)}'\}$ be the family obtained from $\mathcal{F}$ by deleting an arbitrary element from each A_i. Let B be any set such that $B \cap A_i' \neq \Phi$ for $i = 1, 2, \ldots, m(n+1)$. Thus for some j, $1 \leq j \leq m(n+1)$, we have $B \supset A_j \supset A_j'$. Hence $\mathcal{F}'$ does not possess property $\mathcal{B}$. The proof of the theorem is complete.

We proceed to prove (3.2.3). Repeated application of (3.2.4) and the fact that $m(3) = 7$ yields

(3.2.5) $$m(3^k) \leq (\sqrt{7})^{3^k - 1}.$$

For given $\epsilon > 0$, let k be the smallest positive integer such that

(3.2.6) $$1 < \left(\frac{4}{\sqrt{7}}\right)^{3^{-k}} \leq 1 + \frac{\epsilon}{2\sqrt{7}}.$$

Then, if n is of the form $\ell 3^k$, we have

$$
\begin{aligned}
m(n) &= m(\ell 3^k) \\
&\leq m(\ell)m(3^k)^\ell \\
&\leq 4^\ell(\sqrt{7})^{(3^k - 1)\ell} \\
&< (\sqrt{7} + \tfrac{\epsilon}{2})^n ,
\end{aligned}
$$

where we have used (3.2.4), (3.2.1), (3.2.5) and (3.2.6). If $\ell 3^k < n < (\ell + 1)3^k$, we have

$$
\begin{aligned}
m(n) &\leq m((\ell + 1)3^k) \\
&\leq m(\ell + 1)m(3^k)^{\ell+1} \\
&< 4^{\ell+1} (\sqrt{7})^{(3^k - 1)(\ell+1)} \\
&< c_k(\sqrt{7} + \tfrac{\epsilon}{2})^n ,
\end{aligned}
$$

where we have used Theorem 3.2.2, (3.2.4), (3.2.1), (3.2.5) and (3.2.6) and where c_k is a constant depending only on k. Since k is fixed throughout this argument we have

$$m(n) < (\sqrt{7} + \epsilon)^n$$

if n is sufficiently large.

In [5] P. Erdős remarks that it is not known whether or not $\lim_{n \to \infty} m(n)^{1/n}$ exists. We can now settle this question in the affirmative. Let

$$\alpha = \liminf_{n \to \infty} m(n)^{1/n} \leq \limsup_{n \to \infty} m(n)^{1/n} = \beta .$$

Let $\epsilon > 0$ be given. Let n_0 be the smallest positive integer such that $4^{1/n} < 1 + \epsilon$ for $n \geq n_0$. Let $b \geq n_0$ be the smallest integer such that $m(b)^{1/b} < \alpha + \epsilon$. Let $n = ab + r$ where $0 \leq r \leq b - 1$, $a > 1$. (Every sufficiently large integer n can be written in this form.) Some straightforward calculations show that if n is sufficiently large

$$m(n)^{1/n} < \alpha + M\epsilon$$

where M is a constant independent of n and ϵ. It follows that $\alpha = \beta$, as required.

From (3.2.2) and (3.2.3) it follows that

$$2 \leq \lim_{n \to \infty} m(n)^{1/n} \leq \sqrt{7}.$$

The above results are contained in a paper by the author and L. Moser [1]. Since the publication of this paper, P. Erdős [7] has proved that for all n

$$m(n) \leq n^2 2^{n+1},$$

and that for every $\epsilon > 0$ and n sufficiently large,

$$m(n) \leq (1 - \epsilon) n^2 2^{n-1} \log 2.$$

It follows from these results that $\lim_{n \to \infty} m(n)^{1/n} = 2$.

We prove next some recurrence inequalities for $m(n)$ which are not very sharp for large values of n but which yield better results than (3.2.4) when n is small.

<u>Theorem 3.2.3</u> The following inequalities hold

(3.2.7) $m(2k + 1) \leq 2m(k + 1)m(k) + (2k + 1)m(k)^2 + 1$

(3.2.8) $m(3k + 1) \leq 3m(2k + 1)m(k) + (3k + 1)m(k)^3 + 1.$

<u>Proof</u> First we prove (3.2.7). Let

$$\mathcal{G} = \{A_1, A_2, \ldots , A_{m(k+1)}\}$$

$$\mathcal{H} = \{B_1, B_2, \ldots , B_{m(k)}\}$$

$$\mathcal{L} = \{C_1, C_2, \ldots , C_{m(k)}\}$$

be families of sets which do not possess property $\mathcal{B}$. Let $\{a_1, a_2, \ldots , a_{2k+1}\}$ be a set of $2k + 1$ elements. Let $\mathcal{F}$ be the family consisting of the following sets

$$A_i \cup B_j , \quad i = 1, 2, \ldots , m(k + 1), \quad j = 1, 2, \ldots , m(k)$$

$$A_i \cup C_j , \quad i = 1, 2, \ldots , m(k + 1), \quad j = 1, 2, \ldots , m(k)$$

$$B_i \cup C_j \cup [a_\ell] , \quad i,j = 1, 2, \ldots , m(k), \quad \ell = 1, 2, \ldots , 2k + 1$$

$$\{a_1, a_2, \ldots , a_{2k+1}\}.$$

It is clear that the total number of sets in $\mathcal{F}$ is $2m(k + 1)m(k) + (2k + 1)m(k)^2 + 1$ and that each member of $\mathcal{F}$ has $2k + 1$ elements. It remains to be shown that $\mathcal{F}$ does not possess property $\mathcal{B}$. Let B be any set such that $B \cap F \neq \Phi$ for each $F \in \mathcal{F}$. There are three cases to be considered.

<u>Case 1</u> $B \cap B_i = \Phi$ and $B \cap C_j = \Phi$ for some i and j, $1 \leq i \leq m(k)$, $i \leq j \leq m(k)$. Since $B \cap (B_i \cup C_j \cup [a_\ell]) \neq \Phi$ for $\ell = 1, 2, \ldots, 2k + 1$, it follows that $a_\ell \in B$ for $\ell = 1, 2, \ldots, 2k + 1$. Thus $B \supset \{a_1, a_2, \ldots, a_{2k+1}\}$.

<u>Case 2</u> $B \cap B_t = \Phi$ for some t, $1 \leq t \leq m(k)$, and $B \cap C_j \neq \Phi$ for $j = 1, 2, \ldots, m(k)$. Since $\mathcal{L}$ does not possess property $\mathcal{B}$, then for some r, $1 \leq r \leq m(k)$, we have $B \supset C_r$. Since $B \cap (A_i \cup B_t) \neq \Phi$ for $i = 1, 2, \ldots, m(k + 1)$, we must have $B \cap A_i \neq \Phi$ for $i = 1, 2, \ldots, m(k + 1)$. Since $\mathcal{G}$ does not possess property $\mathcal{B}$, it follows that $B \supset A_s$ for some s, $1 \leq s \leq m(k + 1)$. Thus $B \supset (A_s \cup C_r)$.

<u>Case 3</u> $B \cap B_i \neq \Phi$ and $B \cap C_i \neq \Phi$ for $i = 1, 2, \ldots, m(k)$. Then for some t, r, where $1 \leq t, r \leq m(k)$, we have $B \supset B_t$ and $B \supset C_r$. Also since $B \cap \{a_1, a_2, \ldots, a_{2k+1}\} \neq \Phi$ we must have $a_\ell \in B$ for some ℓ, $1 \leq \ell \leq 2k + 1$. Thus $B \supset (B_t \cup C_r \cup [a_\ell])$.

There are no other possibilities and hence the proof of (3.2.7) is complete.

We shall not give all of the details of the proof of (3.2.8). We construct a family of sets in the following manner. Let

$$\begin{aligned}
\mathcal{G} &= \{A_1, A_2, \ldots, A_{m(2k+1)}\} \\
\mathcal{H} &= \{B_1, B_2, \ldots, B_{m(k)}\} \\
\mathcal{L} &= \{C_1, C_2, \ldots, C_{m(k)}\} \\
\mathcal{M} &= \{D_1, D_2, \ldots, D_{m(k)}\}
\end{aligned}$$

be families of sets which do not possess property $\mathcal{B}$, let $\{a_1, a_2, \dots , a_{3k+1}\}$ be a set of $3k + 1$ elements and let $\mathcal{F}$ be the family consisting of the sets

$$A_i \cup B_j, \quad i = 1, 2, \dots , m(2k + 1), \quad j = 1, 2, \dots , m(k)$$

$$A_i \cup C_j, \quad i = 1, 2, \dots , m(2k + 1), \quad j = 1, 2, \dots , m(k)$$

$$A_i \cup D_j, \quad i = 1, 2, \dots , m(2k + 1), \quad j = 1, 2, \dots , m(k)$$

$$B_i \cup C_j \cup D_r \cup [a_\ell], \quad i, j, r, = 1, 2, \dots , m(k), \quad \ell = 1, 2, \dots , 3k + 1$$

$$\{a_1, a_2, \dots , a_{3k+1}\}.$$

It is clear that the total number of sets in $\mathcal{F}$ is $3m(2k + 1)m(k) + (3k + 1)m(k)^3 + 1$ and that each member of $\mathcal{F}$ has $3k + 1$ elements. An argument similar to that used to prove (3.2.7) can now be used to show that $\mathcal{F}$ does not possess property $\mathcal{B}$.

The author was led to (3.2.7) and (3.2.8) while trying to find upper bounds for $m(4)$, $m(5)$ and $m(7)$. From (3.2.8) we get $m(4) \leq 26$ while (3.2.4) with $a = b = 2$ yields $m(4) \leq 27$ and (3.2.1) gives only $m(4) \leq 35$. (3.2.4) cannot be used at all to get upper bounds for $m(5)$ and $m(7)$. (3.2.1) yields $m(5) \leq 126$ and $m(7) \leq 1716$. However, from (3.2.7) we get $m(5) \leq 88$ and $m(7) \leq 708$.

§3.3 A second problem of Erdös and Hajnal

In their paper, Erdös and Hajnal ask whether or not there exists for every positive integer $k \geq 2$ a finite family $\mathcal{F}_k$ of finite sets satisfying

(i) $|F| = k$ for each $F \in \mathcal{F}_k$

(ii) $|F \cap G| \leq 1$ for $F, G \in \mathcal{F}_k$, $F \neq G$

(iii) $\mathcal{F}_k$ does not possess property $\mathcal{B}$?

They observed that such families exist for $k = 2, 3,$. For $k = 2$, one can take

$$\mathcal{F}_2 = \{(1, 2), (1, 3), (2, 3)\}$$

and for $k = 3$, one can take

$$\mathcal{F}_3 = \{(1, 2, 3), (1, 4, 5), (1, 6, 7(, 2, 4, 6), (2, 5, 7), (3, 4, 7), (3, 5, 6)\}.$$

It is not difficult to verify that $\mathcal{F}_2$ and $\mathcal{F}_3$ satisfy conditions (i), (ii) and (iii).

Here we shall show that such families exist for each positive integer $k \geq 2$. In fact, we shall construct such families explicitly. The construction makes use of a special case of Ramsay's Theorem. We prove

Theorem 3.3.1 Let $s \geq N(2, k, r)$ and let S be a set of s elements. Let K be a subset of S with k elements and let F be the set whose

elements are the $\binom{k}{r}$ subsets of K with r elements. Let $\mathcal{F}_{k,r}$ be the family of all possible sets constructed in this way. Then $\mathcal{F}_{k,r}$ does not possess property $\mathcal{B}$.

<u>Proof</u> Assume that $\mathcal{F}_{k,r}$ possesses property $\mathcal{B}$. Then there exists a set $B \subset \bigcup \mathcal{F}_{k,r}$ such that $B \cap F \neq \Phi$ and $B \not\supset F$ for each $F \in \mathcal{F}_{k,r}$. Partition the collection of subsets of S with r elements into two classes $\mathcal{L}_1$ and $\mathcal{L}_2$ by placing an r-subset R of S in $\mathcal{L}_1$ if $R \in B$ and in $\mathcal{L}_2$ if $R \notin B$. Let K be any subset of S with k elements, and let F be the corresponding member of $\mathcal{F}_{k,r}$. Then since $B \cap F \neq \Phi$, there is an r-subset of K which belongs to B and hence to $\mathcal{L}_1$, and since $B \not\supset F$, there is an r-subset of K which does not belong to B and hence belongs to $\mathcal{L}_2$. However, according to Ramsay's Theorem, there must exist some subset of S with k elements all of whose r-subsets belong either to $\mathcal{L}_1$ or to $\mathcal{L}_2$. This is a contradiction and the proof of the theorem is complete.

The question of Erdös and Hajnal can now be settled by observing that $\mathcal{F}_{k,k-1}$ satisfies conditions (i), (ii) and (iii).

If we take $s = N(2, k, r)$ in Theorem 3.3.1, the total number of sets in the family $\mathcal{F}_{k,r}$ is $\binom{N(2,\ k,\ r)}{k}$ and each set has $\binom{k}{r}$ elements. In view of (3.2.2) we must have

$$\binom{N(2,\ k,\ r)}{k} > 2^{\binom{k}{r} - 1}.$$

This result was obtained by Erdös [4] by a different method. In particular if $r = 2$ we have

$$\binom{\ell(2, k)}{k} > 2^{\binom{k}{2} - 1},$$

which is (2.1.4)

§3.4 Property $\mathcal{B}(s)$

Let s be a cardinal number. A family $\mathcal{F}$ of sets is said to possess property $\mathcal{B}(s)$ if there exists a set $B \subset \cup \mathcal{F}$ such that $F \cap B \neq \Phi$ and $|F \cap B| < s$ for every $F \in \mathcal{F}$. Property $\mathcal{B}(s)$ was investigated extensively by Erdös and Hajnal [8] who were primarily interested in infinite families of infinite sets. For finite families of finite sets they asked: If n and s are positive integers with $n \geq s$, what is the smallest positive integer $m(n, s)$ for which there exists a family $\mathcal{F}$ of $m(n, s)$ sets, each set with n elements and which does not possess property $\mathcal{B}(s)$?

It is clear that $m(n, n) = m(n)$ and $m(n, 1) = 1$. Erdös and Hajnal remarked that the family $\mathcal{F}$ consisting of the subsets with n elements of a set of $n + s - 1$ elements does not possess property $\mathcal{B}(s)$ and thus

$$m(n, s) \leq \binom{n + s - 1}{n}. \tag{3.4.1}$$

The upper bound for $m(n, s)$ afforded by (3.4.1) is not very good even for small values of s. For example, if we set $s = 2$ in (3.4.1) we get $m(n, 2) \leq n + 1$. However, as we shall show later,

$m(2k, 2) = 3$ and $m(2k + 1, 2) = 4$.

The main result that we wish to establish in this section is that for all positive integers k and s,

$$m(ks, s) \leq m(s). \tag{3.4.2}$$

Let $\mathcal{F}_s = \{A_1, A_2, \ldots, A_{m(s)}\}$ be a family of sets which does not possess property $\mathcal{B}$. We may assume without loss of generality that, if $|\bigcup \mathcal{F}_s| = \ell$, then the elements of $\bigcup \mathcal{F}_s$ are the positive integers $1, 2, \ldots, \ell$. From A_i we construct a set A_i' as follows: If the elements of A_i are $i_1, i_2, \ldots, i_s$ then the elements of A_i' are those positive integers not exceeding $k\ell$ and which are congruent modulo ℓ to any of $i_1, i_2, \ldots, i_s$. Let $\mathcal{F}$ be the family $\{A_1', A_2', \ldots, A_{m(s)}'\}$. It is clear that each member of $\mathcal{F}$ has ks elements. It remains to be shown that $\mathcal{F}$ does not possess property $\mathcal{B}(s)$. Let $B \subset \bigcup \mathcal{F}$ be such that $B \cap A_i' \neq \Phi$ for $i = 1, 2, \ldots, m(s)$. Thus for each $A \in \mathcal{F}_s$, B contains an element congruent modulo ℓ to some element of A. Since $\mathcal{F}_s$ does not possess property $\mathcal{B}$, there is some $A_i \in \mathcal{F}_s$ such that B contains elements congruent modulo ℓ to each element of A_i. This implies that $|B \cap A_i'| \geq s$; that is, $\mathcal{F}$ does not possess property $\mathcal{B}(s)$.

Various other inequalities can be proved for $m(n, s)$ but we shall not present any of these here.

Finally, we prove that for every positive integer k

$$m(2k,\ 2) = 3$$

$$m(2k + 1,\ 2) = 4 \qquad (3.4.3)$$

$$m(3k,\ 3) = 7\ .$$

These are the only known values of $m(n, s)$. In order to prove (3.4.3) we need

Theorem 3.4.1 Let $\mathcal{F}$ be a family of sets which does not possess property $\mathcal{B}$ (or $\mathcal{B}(s)$). Let $a, b \in \bigcup \mathcal{F}$, $a \neq b$. Then either there is an $F \in \mathcal{F}$ such that $a, b \in F$, or, if this is not the case, the family $\mathcal{F}'$ obtained from $\mathcal{F}$ by replacing b by a in each set in which b occurs does not possess property $\mathcal{B}$ (or $\mathcal{B}(s)$).

The proof of the theorem is easy and will therefore be omitted. The theorem enables us to assert, without loss of generality, that the first possibility given in the conclusion of the theorem always holds.

That $m(2k, 2) \leq 3$ follows from (3.4.2) and it is not difficult to verify that $m(2k, 2) > 2$.

That $m(2k + 1, 2) \leq 4$ follows from the fact that the family $\{A_1 \cup [0],\ A_2 \cup [0],\ A_3 \cup [0],\ A_4\}$, where A_1 is the set of the first $2k$ positive integers congruent to 1 or 2 modulo 3, A_2 is the set of the first $2k$ positive integers congruent to 0 or 1 modulo 3, A_3 is the set of the first $2k$ positive integers congruent to 0 or 2 modulo 3 and $A_4 = \{1, 2, \ldots, 2k + 1\}$, does not possess property $\mathcal{B}(2)$. Let $\mathcal{F} = \{F_1, F_2, F_3\}$ be a family of sets

satisfying $|F_1| = |F_2| = |F_3| = 2k + 1$. Assume that $\mathcal{F}$ does not possess property $\mathcal{B}(2)$. We show that this leads to a contradiction. There are several cases to consider.

Case 1 There is an element a belonging to all three members of $\mathcal{F}$. Then $\mathcal{F}$ clearly has property $\mathcal{B}(2)$.

Case 2 There is an element a which belongs to exactly one member of $\mathcal{F}$, say F_1, Then, by Theorem 3.4.1 we must have $F_2 \subset F_1$ and $F_3 \subset F_1$. This is obviously impossible.

Case 3 Each element of $\bigcup \mathcal{F}$ belongs to exactly two members of $\mathcal{F}$. Then each element of F_1 must occur either in F_2 or F_3, but not both. Let $a_1, a_2, \ldots, a_r$ be the elements of F_1 which belong to F_2 and $a_{r+1}, \ldots, a_{2k+1}$ be the elements which belong to F_3. We may write

$$F_1 = \{a_1, a_2, \ldots, a_r, a_{r+1}, \ldots, a_{2k+1}\}$$

$$F_2 = \{a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_{2k+1-r}\}$$

$$F_3 = \{a_{r+1}, \ldots, a_{2k+1}, c_1, c_2, \ldots, c_r\}.$$

Each element of F_2 must occur either in F_1 or F_3. Thus we must have

$$\{b_1, b_2, \ldots, b_{2k+1-r}\} = \{c_1, c_2, \ldots, c_r\}$$

and hence

$$2k + 1 - r = r.$$

This is also a contradiction. We conclude that $\mathcal{F}$ possesses property $\mathcal{B}(2)$ and hence that $m(2k+1, 2) = 4$.

That $m(3k, 3) \leq 7$ follows from (3.4.2) and the fact that $m(3) = 7$. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_6\}$ be a family of sets satisfying $|F_i| = 3k$ for $i = 1, 2, \ldots, 6$. Assume that $\mathcal{F}$ does not possess property $\mathcal{B}(3)$. We shall show that this assumption leads to a contradiction. There are several cases to be considered.

<u>Case 1</u> There is an element a which belongs to five or more members of $\mathcal{F}$. Then $\mathcal{F}$ clearly possesses property $\mathcal{B}(3)$. Henceforth we assume that no element belongs to more than four members of $\mathcal{F}$.

<u>Case 2</u> There is an element a belonging to four sets, say F_1, F_2, F_3 and F_4. If there is an element b belonging to F_5 and F_6, then $\mathcal{F}$ clearly possesses property $\mathcal{B}(3)$. We may therefore assume $F_5 \cap F_6 = \Phi$. Suppose there is an element $b \in F_5$ which appears in exactly one of the sets F_1, F_2, F_3, F_4, say F_1. Then, by Theorem 3.4.1, we must have $F_6 \subset F_1$. This is impossible. Hence we may assume that each of the elements in F_5 and F_6 belongs to at least two of the sets F_1, F_2, F_3, F_4. However, this also leads to a contradiction since in F_5 and F_6 there are $6k$ distinct elements and in F_1, F_2, F_3, F_4 there are only $12k - 4$ "places" which these elements can occupy. Henceforth we assume that no element belongs to more than three members of $\mathcal{F}$.

Case 3 There is an element a belonging to three sets, say F_1, F_2 and F_3. Suppose $b \in F_4 \cap F_5$. Then, by Theorem 3.4.1 and case 2, b must belong to exactly one of the sets F_1, F_2, F_3, say F_1. Then it is possible to choose $c \in F_6$ different from all of the elements in F_1. It is easy to see that the set $\{a, b, c\}$ has at least one and at most two elements in common with each member of $\mathcal{F}$; that is, $\mathcal{F}$ possesses property $\mathcal{B}(3)$. We may therefore take for granted that F_4, F_5 and F_6 are pairwise disjoint. By Theorem 3.4.1, the $9k$ distinct elements in F_4, F_5, and F_6 must also occur in F_1, F_2 and F_3. However, this is impossible since the sets F_1, F_2 and F_3 together contain at most $9k - 3$ elements different from a.

Henceforth we assume that no element appears in more than two members of $\mathcal{F}$.

Case 4 There is an element a belonging to two sets, say F_1 and F_2. Since we are assuming that no element appears in three or more sets, the sets F_3, F_4, F_5 and F_6 must contain at least $6k$ distinct elements. By Theorem 3.4.1, these must all belong to either F_1 or F_2. This contradicts the fact that F_1 and F_2 together contain at most $6k - 2$ elements different from a.

Case 5 There is an element a which belongs to exactly one member of $\mathcal{F}$, say F_1. Then, by Theorem 3.4.1, we must have $F_2 \subset F_1$. This is clearly impossible.

We conclude that $\mathcal{F}$ possesses property $\mathcal{B}(3)$ and hence that $m(3k, 3) = 7$.

It would be of interest to know whether or not $m(ks, s) = m(s)$ for any other values of s.

§3.5 Some unsolved problems related to property $\mathcal{B}$

In this section we state some unsolved problems and make some brief comments about them.

1. The order of magnitude of $m(n)$ is still not known. Up to the present time it has only been proved that, for all n,

$$2^n\left(\frac{n}{n+4}\right) < m(n) < n^2 2^{n+1} . \tag{3.5.1}$$

One possible approach would be the following: Let $\mathfrak{F} = \{A_1, A_2, \ldots, A_{m(n)}\}$ be a family of sets which does not possess property $\mathcal{B}$. For $a \in \cup \mathfrak{F}$, let $f(a, n)$ denote the number of members of $\mathfrak{F}$ which contain a. If one could obtain information about $f(a, n)$ then this information would give one a deeper insight into the structure of $\mathfrak{F}$ and would possibly lead to some method of improving the bounds for $m(n)$. However, all that the author has been able to prove in this direction is that if $L = \max f(a, n)$, then, for each $a \in \cup \mathfrak{F}$,

$$f(a, n) \geq \frac{nm(n)}{nL - L + n} ,$$

but this does not seem to be very useful. It seems plausible that $f(a, n)$ is independent of a and this is the case when $n = 1, 2, 3$.

2. Let $N \geq 2n - 1$ and denote by $m_N(n)$ the smallest positive integer for which there exists a family of subsets $A_1, A_2, \ldots, A_{m_N(n)}$ of a set of N elements such that $|A_i| = n$ for $i = 1, 2, \ldots, m_N(n)$, and which does not possess property $\mathcal{B}$. What is the order of magnitude of $m_N(n)$? It is not difficult to see that

$$m_{2n-1}(n) = \binom{2n-1}{n},$$

and that if N is sufficiently large

$$m_N(n) = m(n).$$

Let N_0 be the smallest positive integer for which $m_{N_0}(n) = m(n)$. Erdös [7] conjectured that $N_0 \sim cn^2$. This conjecture is supported by the fact that in proving $m(n) < n^2 2^{n+1}$, Erdös chose his family of sets as subsets of a set of $2n^2$ elements.

3. The upper bound for $m(n)$ given by (3.5.1) was obtained by non-constructive arguments. Can one exhibit explicitly a family of $k < cn^2 2^n$ sets, each with n elements, and which does not possess property $\mathcal{B}$?

CHAPTER IV

A PROBLEM OF ERDÖS AND RADO

§4.1 A problem of Erdös and Rado

P. Erdös and R. Rado [9] proved that to each pair of positive integers n and k, with $n \geq 1$ and $k \geq 3$, there corresponds a positive integer $\varphi(n, k)$, which we take to be minimal, such that if $\mathfrak{F}$ is a family of $\ell \geq \varphi(n, k) + 1$ distinct sets, each set with n elements, then some k of these sets have pairwise the same intersection. $\varphi(n, k)$ is thus the largest positive integer m for which there exists a family of m sets, each with n elements, and no k of which have pairwise the same intersection.

Erdös and Rado proved that

$$(4.1.1) \qquad (k-1)^n \leq \varphi(n, k) \leq n!(k-1)^n \left\{1 - \sum_{i=1}^{n-1} \frac{i}{(i+1)!(k-1)^i}\right\}$$

and conjectured that there exists a constant c such that

$$(4.1.2) \qquad \varphi(n, k) < c^n(k-1)^n.$$

The only values of φ which are known are $\varphi(1, k) = k - 1$ and $\varphi(2, 3) = 6$. That $\varphi(2, 3) \leq 6$ follows from (4.1.1) and it is easy to see that in the family $\{(1, 2), (1, 3), (2, 3), (4, 5), (4, 6), (5, 6)\}$ no three sets have pairwise the same intersection. It is also not difficult to verify that no other family of six sets of two elements each has this property.

The main result that we wish to establish in this section is that

$$(4.1.3) \qquad \varphi(n, k) \geq \begin{cases} t^n & , \text{ if } n \text{ is even} \\ (k-1)t^{n-1}, & \text{if } n \text{ is odd,} \end{cases}$$

where

$$t^2 = (k-1)^2 + \left[\frac{k-1}{2} \right].$$

It is clear that the lower bound given by (4.1.3) is better than that given by (4.1.1) for all n and k.

In order to prove (4.1.3) we shall need some preliminary theorems and results.

<u>Theorem 4.1.1</u> For all positive integers a, b and k with $k \geq 3$, we have

$$(4.1.4) \qquad \varphi(a + b, k) \geq \varphi(a, k)\varphi(b, k).$$

<u>Proof</u> Let $\mathcal{F}_a = \{A_1, A_2, \ldots, A_{\varphi(a,k)}\}$ and $\mathcal{F}_b = \{B_1, B_2, \ldots, B_{\varphi(b,k)}\}$ be families of sets having the desired property (that is, no k of the A's and no k of the B's have pairwise the same intersection). As the notation implies, each A has a elements and each B has b elements. We also take for granted that $A_i \cap B_j = \Phi$ for all i and j. Let $\mathcal{F} = \{A_i \cup B_j : i = 1, 2, \ldots, \varphi(a, k), \quad j = 1, 2, \ldots, \varphi(b, k)\}$. It is clear that the number of sets in $\mathcal{F}$ is $\varphi(a, k)\varphi(b, k)$ and that each member of $\mathcal{F}$ has $a + b$ elements. The proof of the theorem will be

complete if we show that no k members of $\mathcal{F}$ have pairwise the same intersection. Suppose there exist distinct sets $F_1, F_2, \ldots, F_k$ in $\mathcal{F}$ and a set $S \subset \bigcup \mathcal{F}$ such that

$$F_i \cap F_j = S, \quad i, j = 1, 2, \ldots, k, \quad i \neq j. \tag{4.1.5}$$

Let $F_i = A_{m_i} \cup B_{n_i}$ for $i = 1, 2, \ldots, k$. Partition the elements of S into two sets S_1 and S_2, an element being placed in S_1 if it belongs to $\bigcup \mathcal{F}_a$ and in S_2 if it belongs to $\bigcup \mathcal{F}_b$. Then it is not difficult to see, using (4.1.5), that

$$A_{m_i} \cap A_{m_j} = S_1, \quad i, j = 1, 2, \ldots, k, \quad i \neq j \tag{4.1.6}$$

and

$$B_{n_i} \cap B_{n_j} = S_2, \quad i, j = 1, 2, \ldots, k, \quad i \neq j. \tag{4.1.7}$$

If the sets $A_{m_1}, A_{m_2}, \ldots, A_{m_k}$ are all distinct or if the sets $B_{n_1}, B_{n_2}, \ldots, B_{n_k}$ are all distinct then we have a contradiction. Hence two of the A_{m_i} must be identical and two of the B_{n_i} must be identical. Thus, in view of (4.1.6) and (4.1.7), we have

$$A_{m_1} = A_{m_2} = \ldots = A_{m_k}$$

and

$$B_{n_1} = B_{n_2} = \ldots = B_{n_k}.$$

Hence

$$F_1 = F_2 = \ldots = F_k.$$

This contradicts the fact that the F's were chosen as distinct subsets of $\mathfrak{F}$. The proof of the theorem is complete.

If we set $b = a$ in (4.1.4) we get

$$\varphi(2a, k) \geq \varphi(a, k)^2,$$

and this implies that for each positive integer m

$$\varphi(2^m, k) \geq \varphi(2, k)^{2^{m-1}}. \tag{4.1.8}$$

Let $n = 2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_t}$, where $\alpha_1 > \alpha_2 > \ldots > \alpha_t \geq 0$, be the binary representation of n. If n is even, we get from (4.1.4) and (4.1.8),

$$\begin{aligned}\varphi(n, k) &= \varphi(2^{\alpha_1} + 2^{\alpha_2} + \ldots + 2^{\alpha_t}, k) \\ &\geq \varphi(2^{\alpha_1}, k)\varphi(2^{\alpha_2}, k) \ldots \varphi(2^{\alpha_t}, k) \\ &\geq \varphi(2, k)^{2^{\alpha_1 - 1} + 2^{\alpha_2 - 1} + \ldots + 2^{\alpha_t - 1}} \\ &= \varphi(2, k)^{n/2}.\end{aligned}$$

A similar argument shows that, if n is odd, then

$$\begin{aligned}\varphi(n, k) &\geq \varphi(1, k)\varphi(2, k)^{\frac{n-1}{2}} \\ &= (k - 1)\varphi(2, k)^{\frac{n-1}{2}}.\end{aligned}$$

Hence

$$\varphi(n, k) \geq \begin{cases} \varphi(2, k)^{n/2}, & \text{if } n \text{ is even} \\ (k - 1)\varphi(2, k)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases} \tag{4.1.9}$$

We turn our attention now to the derivation of a lower bound for $\varphi(2, k)$. We prove

Theorem 4.1.2

$$\varphi(2, k) \geq (k - 1)^2 + \left[\frac{k - 1}{2} \right]. \tag{4.1.10}$$

Proof Let $N = \{1, 2, \dots, 2k - 1\}$. Suppose first that k is odd and let $\ell = (k - 1)/2$. We show how to select $(k - 1)^2 + \ell$ subsets of N, each set with two elements, no k of which have pairwise the same intersection. Let

$$\mathcal{F}_1 = \{(i, j) : i = 1, 2, \dots, \ell, \quad j = k + 1, \dots, 2k - 1\}$$

$$\mathcal{F}_2 = \{(i, j) : i = \ell + 1, \dots, k - 1, \quad j = k + \ell - 1, \dots, 2k - 1\}$$

$$\mathcal{F}_3 = \{(i, j) : i = \ell + 1, \dots, k - 1, \quad j = \ell + 2, \dots, k, \quad i < j\}$$

$$\mathcal{F}_4 = \{(i, j) : i = k, \dots, k + \ell - 1, \quad j = k + 1, \dots, k + \ell, \quad i < j\}.$$

It is not difficult to check that the families $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ are pairwise disjoint and that we have

$$|\mathcal{F}_1| = \ell(k - 1)$$

$$|\mathcal{F}_2| = (k - \ell)(k - \ell - 1)$$

and

$$|\mathcal{F}_3| = |\mathcal{F}_4| = \frac{\ell(\ell + 1)}{2}.$$

Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$. Then

$$\begin{aligned} |\mathcal{F}| &= |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| + |\mathcal{F}_4| \\ &= \ell(k - 1) + (k - \ell)(k - \ell - 1) + \ell(\ell + 1) \\ &= (k - 1)^2 + \ell. \end{aligned}$$

One can readily convince oneself that each of $1, 2, \ldots, 2k - 1$ appears in exactly $k - 1$ members of $\mathcal{F}$. Thus if k members of $\mathcal{F}$ are to have pairwise the same intersection, they must be pairwise disjoint. But this contradicts the fact that $|\bigcup \mathcal{F}| = |N| = 2k - 1$. It follows that no k members of $\mathcal{F}$ have pairwise the same intersection.

Consider next the case where k is even and set $\ell = k/2$. We show how to construct $(k - 1)^2 + \ell - 1$ sets, each with two elements, no k of which have pairwise the same intersection. Let

$$\mathcal{F}_1 = \{(i, j) : i = 1, 2, \ldots, \ell, \;\; j = k + 1, \ldots, 2k - 1\}$$

$$\mathcal{F}_2 = \{(i, j) : i = \ell + 1, \ldots, k - 1, \;\; j = k + \ell, \ldots, 2k - 1\}$$

$$\mathcal{F}_3 = \{(i, j) : i = \ell + 1, \ldots, k - 1, \;\; j = \ell + 2, \ldots, k, \;\; i < j\}$$

$$\mathcal{F}_4 = \{(i, j) : i = k, \ldots, k + \ell - 2, \;\; j = k + 1, \ldots, k + \ell - 1, \;\; i < j\}.$$

Then $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ are pairwise disjoint and

$$|\mathcal{F}_1| = \ell(k - 1)$$

$$|\mathcal{F}_2| = (k - \ell)(k - \ell - 1)$$

and

$$|\mathcal{F}_3| = |\mathcal{F}_4| = \frac{\ell(\ell - 1)}{2} .$$

Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$. We have

$$\begin{aligned} |\mathcal{F}| &= \ell(k - 1) + (k - \ell)(k - \ell + 1) + \ell(\ell - 1) \\ &= (k - 1)^2 + \ell - 1 . \end{aligned}$$

It is not difficult to verify that no k of the sets in $\mathcal{F}$ have pairwise the same intersection. This completes the proof of Theorem 4.1.2.

It follows from (4.1.9) and (4.1.10) that (4.1.3) holds.

For the convenience of the reader, we illustrate Theorem 4.1.2 in the case $k = 5$. If $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, we may represent the family of all 2-subsets of N in an array.

(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 8)	(1, 9)
	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 7)	(2, 8)	(2, 9)
		(3, 4)	(3, 5)	(3, 6)	(3, 7)	(3, 8)	(3, 9)
			(4, 5)	(4, 6)	(4, 7)	(4, 8)	(4, 9)
				(5, 6)	(5, 7)	(5, 8)	(5, 9)
					(6, 7)	(6, 8)	(6, 9)
						(7, 8)	(7, 9)
							(8, 9)

The families $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ are enclosed inside the heavy lines. It is easy to see from the above array that each of the

elements of N appear in exactly 4 of the sets in $\mathcal{F}$ and that no five of the sets have pairwise the same intersection.

§4.2 A lower bound for $\varphi(n, 3)$

In this section we discuss briefly what is perhaps the most interesting special case of the Erdős-Rado problem, namely, the case where $k = 3$. It is not difficult to verify that among the following sets, no three have pairwise the same intersection:

$$\begin{array}{llll} (1, 2, 7), & (1, 3, 7), & (2, 3, 7), & (4, 5, 8), \\ (4, 6, 8), & (5, 6, 8), & (1, 2, 9), & (1, 3, 9), \\ (2, 3, 9), & (4, 5, 10), & (4, 6, 10), & (5, 6, 10), \\ (7, 8, 9), & (7, 8, 10), & (7, 9, 10), & (8, 9, 10). \end{array}$$

Thus

$$\varphi(3, 3) \geq 16.$$

From (4.1.4) it follows that

$$\begin{aligned} \varphi(3m, 3) &\geq \varphi(3, 3)^m \\ \varphi(3m + 1, 3) &\geq 2\varphi(3, 3)^m \\ \varphi(3m + 2, 3) &\geq 6\varphi(3, 3)^m. \end{aligned}$$

Hence

$$\varphi(n, 3) > c(16)^{n/3}. \tag{4.2.1}$$

This lower bound for $\varphi(n, 3)$ is better than the one afforded by (4.1.3).

§4.3 An application to a problem in number theory

In [6], P. Erdős considered the following problem: What is the largest positive integer $f(n)$ for which there exists a sequence of integers $a_1, a_2, \dots, a_{f(n)}$ satisfying

(i) $1 \leq a_1 < a_2 < \dots < a_{f(n)} \leq n$

(ii) no three of the a's have pairwise the same greatest common divisor?

This question is closely related to the problem of Erdős and Rado. Suppose $\{A_1, A_2, \dots, A_k\}$ is a family of finite sets no three of which have pairwise the same intersection. Let $\bigcup_{i=1}^{k} A_i = \{a_1, a_2, \dots, a_\ell\}$. Let $p_1, p_2, \dots, p_\ell$ be a set of distinct primes and form the numbers $N_1, N_2, \dots, N_k$ where $N_r = \prod_{a_i \in A_r} p_i$. Then it is clear that no three of the N's have pairwise the same greatest common divisor.

Consider the first $10k$ primes and arrange these in an array

$$A = \begin{bmatrix} p_1^1 & p_1^2 & p_1^3 & \dots & p_1^k \\ p_2^1 & p_2^2 & p_2^3 & \dots & p_2^k \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ p_{10}^1 & p_{10}^2 & p_{10}^3 & \dots & p_{10}^k \end{bmatrix}$$

From the primes in the j^{th} column of A we form the numbers

$$\begin{array}{llll}
b_1^j = p_1^j\, p_2^j\, p_7^j & b_5^j = p_4^j\, p_6^j\, p_8^j & b_9^j = p_2^j\, p_3^j\, p_9^j & b_{13}^j = p_7^j\, p_8^j\, p_9^j \\
b_2^j = p_1^j\, p_3^j\, p_7^j & b_6^j = p_5^j\, p_6^j\, p_8^j & b_{10}^j = p_4^j\, p_5^j\, p_{10}^j & b_{14}^j = p_7^j\, p_8^j\, p_{10}^j \\
b_3^j = p_2^j\, p_3^j\, p_7^j & b_7^j = p_1^j\, p_2^j\, p_9^j & b_{11}^j = p_4^j\, p_6^j\, p_{10}^j & b_{15}^j = p_7^j\, p_9^j\, p_{10}^j \\
b_4^j = p_4^j\, p_5^j\, p_8^j & b_8^j = p_1^j\, p_3^j\, p_9^j & b_{12}^j = p_5^j\, p_6^j\, p_{10}^j & b_{16}^j = p_8^j\, p_9^j\, p_{10}^j .
\end{array}$$

Now form the set S_k of the 16^k numbers

$$b_{i_1}^1 \quad b_{i_2}^2 \quad \cdots \quad b_{i_k}^k$$

where $i_1, i_2, \ldots, i_k$ take on the values $1, 2, \ldots, 16$. Then, in view of the remark made earlier and the results given in section 4.2, no three of the numbers in S_k have pairwise the same greatest common divisor. (This can also be established by a straight forward induction on k.)

Let $\epsilon > 0$ be given and choose

$$(4.3.1) \qquad k = \left[\frac{\log n}{(3+\epsilon)\log\log n} \right] .$$

We show that the numbers in S_k do not exceed n, if n is sufficiently large. Each number in S_k is the product of $3k$ distinct primes, none of which exceeds the $10k^{th}$ prime. Hence the largest number in S_k does not exceed

$$\prod_{p_{7k} < p \le p_{10k}} p$$

where p_m denotes the m^{th} prime. The prime number theorem and some

straightforward calculation shows that, if n is sufficiently large and k is defined by (4.3.1), then

$$\prod_{p_{7k} < p \leq p_{10k}} p < n.$$

We therefore have

$$f(n) > 16^{\frac{\log n}{(3 + \epsilon)\log \log n}}$$

for every $\epsilon > 0$ and $n \geq n_o(\epsilon)$.

Using the fact that

$$\varphi(\ell, 3) < \ell!\, 2^{\ell}$$

Erdös [7] was able to prove that

$$f(n) < n^{3/4 + \epsilon}$$

for every $\epsilon > 0$ and n sufficiently large. If one could prove that

$$\varphi(\ell, 3) < c^{\ell}$$

then it would follow that

$$f(n) < c^{\frac{\log n}{\log \log n}}.$$

We mention in conclusion that for the more general problem in which we insist that no t of the a's have pairwise the same greatest common divisor, it can be shown (defining $f(n, t)$ in an obvious way) that

$$f(n, t) > \varphi(2, t)^{\frac{\log n}{(2 + \epsilon) \log \log n}}$$

for every $\epsilon > 0$ and every fixed t, provided $n > n_0(t, \epsilon)$.

§4.4 A relationship between the problem of Erdös and Rado and Ramsay's Theorem

We conclude this chapter by indicating that there is a relationship between the problem of Erdös and Rado and a special case of Ramsay's Theorem discussed in Chapter II.

Let $\{A_1, A_2, \ldots, A_{\varphi(n,k)}\}$ be a family of sets such that each set has n elements and no k of the sets have pairwise the same intersection. Form all possible intersections $A_i \cap A_j$, $i \neq j$, and denote the family of distinct intersections by $\{I_1, I_2, \ldots, I_t\}$. Let G be a complete graph with vertices $P_1, P_2, \ldots, P_{\varphi(n,k)}$. Color the edges of G in t colors $C_1, C_2, \ldots, C_t$ by coloring the edge joining P_i and P_j color C_r if $A_i \cap A_j = I_r$. A simple argument shows that G contains no complete subgraph on k vertices all of whose edges have the same color. Note that we have

$$h(t, k) \geq \varphi(n, k) \tag{4.4.1}$$

where h is the function defined in connection with the special case of Ramsay's Theorem discussed in Chapter II. However, since we do not have any estimates for t in terms of n and k, (4.4.1) does not seem to be of any use at the present time.

CHAPTER V

PATHS ON THE n-CUBE

§5.1 Snakes

For $n \geq 2$, let C_n denote the set of the 2^n vertices $(\delta_1, \delta_2, \ldots, \delta_n)$, where $\delta_i = 0, 1$ for $i = 1, 2, \ldots, n$, of an n-dimensional unit cube. Call two points of C_n adjacent if the distance between them is 1. By a snake of length m in C_n we mean a sequence $\{P_1, P_2, \ldots, P_m\}$ of $m \geq 4$ distinct points of C_n with the following properties:

(i) P_i and P_{i+1} are adjacent for $i = 1, 2, \ldots, m - 1$

(ii) P_1 and P_m are adjacent

(iii) no P_i is adjacent to more than two points of the sequence.

It is clear that these conditions imply that each point of the snake is adjacent to exactly two other points of the snake.

Throughout this section we shall need the following two theorems which are easy to prove.

Theorem 5.1.1 If $\{P_1, P_2, \ldots, P_m\}$ is a snake in C_n, then m is even.

Theorem 5.1.2 If $\{P_1, P_2, \ldots, P_m\}$ is a snake in C_n and if $\{Q_1, Q_2, \ldots, Q_\ell\}$ is a proper subsequence of $\{P_1, P_2, \ldots, P_m\}$, then $\{Q_1, Q_2, \ldots, Q_\ell\}$ is not a snake.

Denote by $s(n)$ the largest positive integer m for which there exists a snake of length m in C_n. It is known that $s(2) = 4$, $s(3) = 6$, $s(4) = 8$ and $s(5) = 14$. The sequences

$\{(0,0), (1,0), (1,1), (0,1)\}$,

$\{(0,0,0), (1,0,0), (1,0,1), (1,1,1), (0,1,1), (0,1,0)\}$,

$\{(0,0,0,0), (1,0,0,0), (1,0,1,0), (1,1,1,0), (1,1,1,1), (0,1,1,1), (0,1,0,1), (0,0,0,1)\}$,

and

$\{(0,0,0,0,0), (1,0,0,0,0), (1,0,1,0,0), (1,1,1,0,0), (0,1,1,0,0), (0,1,1,1,0), (0,1,0,1,0), (1,1,0,1,0), (1,1,0,1,1), (1,1,1,1,1), (1,0,1,1,1), (0,0,1,1,1), (0,0,1,0,1), (0,0,0,0,1)\}$

are snakes of length 4, 6, 8 and 14 in C_2, C_3, C_4 and C_5 respectively. By trial and error, it can be shown that there are no longer snakes in these cubes. The value of $s(n)$ is not known for $n \geq 6$.

W. Kautz [23] proved that

$$s(n+2) \geq 2s(n),$$

and hence

$$s(n) > c2^{n/2}.$$

In [15], it is proved that

$$(5.1.1) \qquad s(n+1) \geq \begin{cases} 3/2\, s(n) & \text{if } s(n) \equiv 0 \pmod 4 \\ \\ 3/2\, s(n) - 1 & \text{if } s(n) \equiv 2 \pmod 4 \end{cases}$$

From (5.1.1) it follows that

(5.1.2) $$s(n) > c(3/2)^n.$$

We prove a recurrence inequality for $s(n)$ which yields a better lower bound than the one given by (5.1.2).

Theorem 5.1.3 For all positive integers $n \geq 2$

(5.1.3) $$s(n+2) \geq \begin{cases} 5/2\ s(n) & \text{if } s(n) \equiv 0 \pmod 8 \\ 5/2\ s(n) - 3 & \text{if } s(n) \equiv 2 \pmod 8 \\ 5/2\ s(n) - 6 & \text{if } s(n) \equiv 4 \pmod 8 \\ 5/2\ s(n) - 9 & \text{if } s(n) \equiv 6 \pmod 8. \end{cases}$$

Note: Throughout the remainder of this section we shall be using the following notation: If $P = (\delta_1, \delta_2, \ldots, \delta_a)$ is a point in C_a and $Q = (\delta_1', \delta_2', \ldots, \delta_b')$ is a point in C_b, then $P + Q$ denotes the point $(\delta_1, \delta_2, \ldots, \delta_a, \delta_1', \delta_2', \ldots, \delta_b')$ in C_{a+b}.

Proof of Theorem 5.1.3: Let $\{P_1, P_2, \ldots, P_{s(n)}\}$ be a snake in C_n. Suppose that $s(n) \equiv 0 \pmod 8$ and let $s(n) = 8k$. We shall construct a snake of length $20k = 5/2\ s(n)$ in C_{n+2}. In fact, the following sequence of points is such a snake:

$$\begin{array}{l}
\{P_1 + (0,0),\ P_1 + (1,0),\ P_1 + (1,1),\ P_2 + (1,1),\ P_2 + (0,1),\\
\quad P_3 + (0,1),\ P_3 + (0,0),\ P_3 + (1,0),\ P_4 + (1,0),\ P_4 + (1,1),\\
\quad P_5 + (1,1),\ P_5 + (0,1),\ P_5 + (0,0),\ P_6 + (0,0),\ P_6 + (1,0),\\
\quad P_7 + (1,0),\ P_7 + (1,1),\ P_7 + (0,1),\ P_8 + (0,1),\ P_8 + (0,0),\\
\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
P_{8k-7} + (0,0),\ P_{8k-7} + (1,0),\ P_{8k-7} + (1,1),\ P_{8k-6} + (1,1),\ P_{8k-6} + (0,1),\\
P_{8k-5} + (0,1),\ P_{8k-5} + (0,0),\ P_{8k-5} + (1,0),\ P_{8k-4} + (1,0),\ P_{8k-4} + (1,1),\\
P_{8k-3} + (1,1),\ P_{8k-3} + (0,1),\ P_{8k-3} + (0,0),\ P_{8k-2} + (0,0),\ P_{8k-2} + (1,0),\\
P_{8k-1} + (1,0),\ P_{8k-1} + (1,1),\ P_{8k-1} + (0,1),\ P_{8k} + (0,1),\ P_{8k} + (0,0)\}.
\end{array}$$

The other parts of the recurrence inequalities can be established in a similar fashion. The proofs will be omitted.

From (5.1.3) it follows that

(5.1.4) $$s(n) > c(5/2)^{n/2}.$$

Since $\sqrt{5/2} = 1.58\ldots$, (5.1.4) is stronger than (5.1.2).

We discuss next the problem of finding upper bounds for $s(n)$. In [15] it is proved that, for $n \geq 3$,

$$s(n) \leq 3.2^{n/2}.$$

This upper bound for $s(n)$ is quite easy to obtain. Let P be a point in C_{n-2}. Since $\{(0,0), (1,0), (1,1), (0,1)\}$ is a snake in C_2, at most three of the points $P + (0,0)$, $P + (1,0)$, $P + (1,1)$, $P + (0,1)$

belong to the longest snake in C_n, by Theorem 5.1.2. There are 2^{n-2} choices for P, and thus in the longest snake in C_n there are at most 3.2^{n-2} points.

Here we shall sketch a proof of

$$(5.1.5) \qquad s(n) \leq 2^{n-1},$$

for $n \geq 4$. Since (5.1.5) is true for $n = 4, 5$, we may take for granted that $n \geq 6$. Let P be in C_{n-5} and consider the set of points $\{P + Q : Q \in C_5\}$ in C_n. It can be shown that among any seventeen vertices of C_5 there is one vertex which is adjacent to at least three others. Thus at most sixteen of the points $\{P + Q : Q \in C_5\}$ belong to the longest snake in C_n. Since there are 2^{n-5} choices for P, the longest snake in C_n can have at most $16.2^{n-5} = 2^{n-1}$ points. This proves (5.1.5).

It would be of interest to know whether nor not $s(n) = o(2^n)$.

While the upper and lower bounds that we have given for $s(n)$ are quite far apart, we are still able to gain a little more insight into the behaviour of $s(n)$. We prove a theorem from which we deduce that the sequence $\{s(n)^{1/n}\}$ converges.

<u>Theorem 5.1.4</u> For all positive integers a and b with $a \geq 2$ and $b \geq 2$,

$$(5.1.6) \qquad s(a+b) \geq \begin{cases} \dfrac{s(a)s(b)}{2} - s(b) + 2, & \text{if } s(a) \equiv 2 \pmod 4 \\[2ex] \dfrac{s(a)s(b)}{2}, & \text{if } s(a) \equiv 0 \pmod 4. \end{cases}$$

<u>Proof</u> Let $\{A_1, A_2, \ldots, A_{s(a)}\}$ and $\{B_1, B_2, \ldots, B_{s(b)}\}$ be snakes in C_a and C_b, respectively. Suppose first that $s(a) \equiv 2 \pmod 4$ and consider the following sequence of points in C_{a+b}:

$$\begin{array}{lllll}
\{A_2 + B_{s(b)-1}, & A_3 + B_{s(b)-1}, & A_3 + B_{s(b)-2}, & \ldots, & A_3 + B_1, \\
A_4 + B_1, & A_5 + B_1, & A_5 + B_2, & \ldots, & A_5 + B_{s(b)-1}, \\
A_6 + B_{s(b)-1}, & A_7 + B_{s(b)-1}, & A_7 + B_{s(b)-2}, & \ldots, & A_7 + B_1, \\
A_8 + B_1, & A_9 + B_1, & A_9 + B_2, & \ldots, & A_9 + B_{s(b)-1}, \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
A_{s(a)-2} + B_1, & A_{s(a)-1} + B_1, & A_{s(a)-1} + B_2, & \ldots, & A_{s(a)-1} + B_{s(b)-1}, \\
A_{s(a)} + B_{s(b)-1}, & A_1 + B_{s(b)-1}\}. & & &
\end{array}$$

It is not difficult to verify that this sequence is a snake of length $\frac{s(a)s(b)}{2} - s(b) + 2$ in C_{a+b}. If $s(a) \equiv 0 \pmod 4$, the following sequence of points is a snake of length $\frac{s(a)s(b)}{2}$ in C_{a+b}:

$$\begin{array}{lllll}
\{A_1 + B_1, & A_1 + B_2, & \ldots, & A_1 + B_{s(b)-1}, & A_2 + B_{s(b)-1}, \\
A_3 + B_{s(b)-1}, & A_3 + B_{s(b)-2}, & \ldots, & A_3 + B_1, & A_4 + B_1, \\
A_5 + B_1, & A_5 + B_2, & \ldots, & A_5 + B_{s(b)-1}, & A_6 + B_{s(b)-1}, \\
A_7 + B_{s(b)-1}, & A_7 + B_{s(b)-2}, & \ldots, & A_7 + B_1, & A_8 + B_1, \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
A_{s(a)-1} + B_{s(b)-1}, & A_{s(a)-1} + B_{s(b)-2}, & \ldots, & A_{s(a)-1} + B_1, & A_{s(a)} + B_1\}.
\end{array}$$

The proof the the theorem is complete.

Note that, in any case, we have

$$(5.1.7) \qquad s(a+b) \geq \frac{s(a)s(b)}{2} - s(b) + 2.$$

From (5.1.7) it follows easily that, for $b \geq 2$ and all positive integers k,

$$(5.1.8) \qquad s(kb) \geq \frac{s(b)^k}{2^k} .$$

Let

$$\alpha = \liminf_{n \to \infty} s(n)^{1/n} \leq \limsup_{n \to \infty} s(n)^{1/n} = \beta .$$

Let $\epsilon > 0$ be given. Determine the smallest positive integers b and ℓ such that $s(b)^{1/b} > \beta - \epsilon$, $2^{1/b} < 1 + \epsilon$ and $\beta^{-1/\ell} > 1 - \epsilon$. Let $n = kb + t$ where $k \geq \ell$ and $0 \leq t \leq b - 1$. Every sufficiently large n can be written in this form. Then (5.1.8) and some straightforward calculations show that

$$s(n)^{1/n} > \beta - M\epsilon$$

where M is a constant independent of n and ϵ. It follows that $\alpha = \beta$ and hence that $\lim_{n \to \infty} s(n)^{1/n}$ exists. The value of $\ell = \lim_{n \to \infty} s(n)^{1/n}$ is not known. However, it follows from (5.1.4) and (5.1.5) that $\sqrt{5/2} \leq \ell \leq 2$.

§5.2 Hamiltonian cycles in the n-cube

By a Hamiltonian cycle in C_n is meant a sequence $\{P_1, P_2, \ldots, P_{2^n}\}$ of distinct vertices of C_n such that

(i) P_i and P_{i+1} are adjacent for $i = 1, 2, \ldots, 2^n-1$

(ii) P_1 and P_{2^n} are adjacent.

That Hamiltonian cycles exist in C_n for every n can be proved easily by induction on n. When $n = 2$, it is clear that $\{(0,0), (1,0), (1,1), (0,1)\}$ is a Hamiltonian cycle. Let $\{P_1, P_2, \ldots, P_{2^n}\}$ be a Hamiltonian cycle in C_n. Then it is clear that $\{P_1+(0), P_2+(0), \ldots, P_{2^n}+(0), P_{2^n}+(1), \ldots, P_2+(1), P_1+(1)\}$ is a Hamiltonian cycle in C_{n+1}.

We shall not regard the Hamiltonian cycles $\{P_1, P_2, \ldots, P_{2^n}\}$ and $\{P_{2^n}, P_{2^n-1}, \ldots, P_1\}$ as being different, nor shall we regard any two cycles obtained from either one of these by cyclic permutation of the points as being different.

Denote by $h(n)$ the number of different Hamiltonian cycles in C_n. The value of $h(n)$ is known only for $n = 2, 3$. It is clear that the only Hamiltonian cycle in C_2 is $\{(0,0), (1,0), (1,1), (0,1)\}$ and thus $h(2) = 1$. By trial and error, it can be shown that the only Hamiltonian cycles in C_3 are

$\{(0,0,0), (1,0,0), (1,0,1), (1,1,1), (1,1,0), (0,1,0), (0,1,1), (0,0,1)\}$

$\{(0,0,0), (1,0,0), (1,0,1), (0,0,1), (0,1,1), (1,1,1), (1,1,0), (0,1,0)\}$

$\{(0,0,0), (1,0,0), (1,1,0), (0,1,0), (0,1,1), (1,1,1), (1,0,1), (0,0,1)\}$

$\{(0,0,0), (1,0,0), (1,1,0), (1,1,1), (1,0,1), (0,0,1), (0,1,1), (0,1,0)\}$

$\{(0,0,0), (0,1,0), (0,1,1), (1,1,1), (1,1,0), (1,0,0), (1,0,1), (0,0,1)\}$

$\{(0,0,0), (0,1,0), (1,1,0), (1,0,0), (1,0,1), (1,1,1), (0,1,1), (0,0,1)\}$.

Thus $h(3) = 6$.

E. N. Gilbert [10] constructed a large number of non-equivalent Hamiltonian cycles in C_n. (Two cycles are equivalent of one can be changed into the other by applying to C_n one of the symmetry operations of the group of symmetries of C_n.) Gilbert's lower bound for the number of non-equivalent Hamiltonian cycles in C_n is $c(n - 2)!\, n^{-1/2}$.

Here we shall obtain a recurrence inequality for $h(n)$ and from this deduce a lower bound.

<u>Theorem 5.2.1</u> For all positive integers $n, m \geq 2$ we have

$$h(n + m) \geq n2^{n-1}\left(\frac{2h(n)}{n}\right)^{2^m} h(m). \tag{5.2.1}$$

<u>Proof</u> Let e be an edge of C_n and denote by T_e the number of Hamiltonian cycles which traverse e. (A Hamiltonian cycle traverses e if the end points of e are consecutive points of the cycle.) For each edge e of C_n the following equality holds:

$$T = T_e = \frac{2h(n)}{n}. \tag{5.2.2}$$

By symmetry, the number of Hamiltonian cycles traversing any one edge is the same as the number traversing any other. Since the number of edges of C_n is $n2^{n-1}$, the total number of Hamiltonian cycles (counting multiplicities) is $Tn2^{n-1}$. Each cycle is counted with 2^n edges. Thus the total number of different Hamiltonian cycles is $\frac{Tn2^{n-1}}{2^n} = \frac{Tn}{2}$. By definition, the total number of different Hamiltonian cycles is $h(n)$. It follows that (5.1.2) holds.

Let e be an edge in C_n and let the end points of e be P and Q. For $i = 1, 2, \ldots, T$ let $\mathcal{P}_i = \{P, P_{i1}, P_{i2}, \ldots, P_{is}, Q\}$ be a Hamiltonian cycle which traverses e. $(s = 2^n - 2)$. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_{2^m}\}$ be a Hamiltonian cycle in C_m. It is not difficult to show that the following sequence of points is a Hamiltonian cycle in C_{n+m}.

$$
\begin{array}{l}
\{P_1 + P,\ P_1 + P_{11},\ P_1 + P_{12},\ P_1 + P_{13},\ \ldots,\ P_1 + P_{1s},\ P_1 + Q, \\
P_2 + Q,\ P_2 + P_{2s},\ P_2 + P_{2,s-1},\ P_2 + P_{2,s-2},\ \ldots,\ P_2 + P_{21},\ P_2 + P, \\
P_3 + P,\ P_3 + P_{31},\ P_3 + P_{32},\ P_3 + P_{33},\ \ldots,\ P_3 + P_{3s},\ P_3 + Q, \\
\quad \vdots \\
P_{2^m} + Q,\ P_{2^m} + P_{2^m,s},\ P_{2^m} + P_{2^m,s-1},\ \ldots,\ P_{2^m} + P_{2^m,1},\ P_{2^m} + P\}.
\end{array}
$$

Now each $\mathcal{P}_i$ can be chosen in T ways, $\mathcal{P}$ in $h(m)$ ways and e can be chosen in $n.2^{n-1}$ ways. The total number of Hamiltonian cycles that can be constructed in the above manner is therefore $n2^{n-1}\,T^{2^m}h(m)$. This, with (5.2.2), implies (5.2.1).

If we set $n = 3$ in (5.2.1) and use the fact that $h(3) = 6$ we get

$$(5.1.3) \qquad h(m + 3) \geq 12(4)^{2^m} h(m),$$

and from (5.1.3) it follows that

$$(5.1.4) \qquad h(n) > (\sqrt{7}/4)^{2^n} .$$

Using (5.1.1) it can be shown that $\ell = \lim_{n \to \infty} h(n)^{2^{-n}}$ exists, but we cannot decide whether the limit is finite or infinite, although it seems likely that $\ell = \infty$.

Instead of Theorem 5.1.1, we could have proved the following: Let n and m be positive integers with $n,m \geq 2$. In C_n let P and Q be points of opposite parity, that is, the number of coordinates in which P differs from Q is odd. Denote by $T(P, Q)$ the number of Hamiltonian paths with end points P and Q (a Hamiltonian path in C_n is a sequence of 2^n distinct points of C_n in which consecutive points are adjacent). Let $T_n^* = \max T(P,Q)$, where the points P and Q range over all pairs of points of C_n of opposite parity. Then it can be shown that

$$h(n + m) \geq (T_n^*)^{2^m} h(m) .$$

It is not difficult to verify that in C_3 there are six Hamiltonian paths with end points (0,0,0) and (1,1,1). Thus

$$h(m + 3) \geq 6^{2^m} h(m),$$

so that

$$h(n) > (\sqrt[7]{6})^{2^n} .$$

In conclusion we remark that the number of non-equivalent Hamiltonian cycles in C_n is at least $\frac{h(n)}{n!\,2^n}$. Thus we have a lower bound for the number of non-equivalent Hamiltonian cycles which is considerably larger than the one obtained by Gilbert [10].

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